title

The Topology of Neuronal Firing Rate Space

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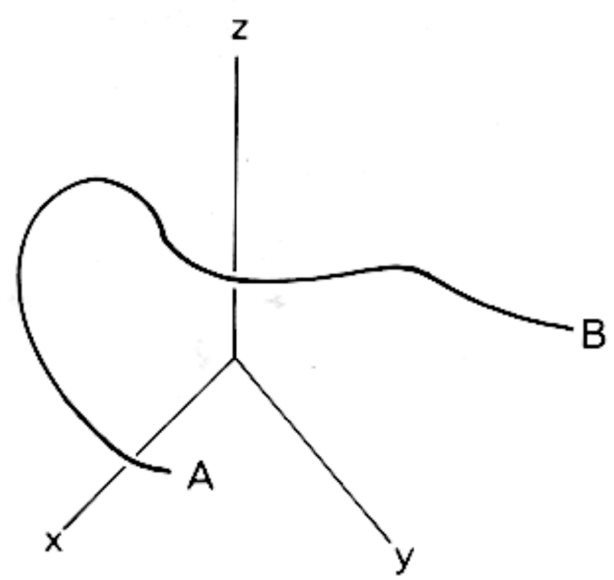
Definition of a dimension

A dimension is something that changes continuously. In most applications to brain it is movement in space and changes in intensity.

Many things that are called dimensions in psychology and statistics are discontinuous and are not considered as dimensions here. Using this definition of dimension may be the hardest part of understanding this approach.

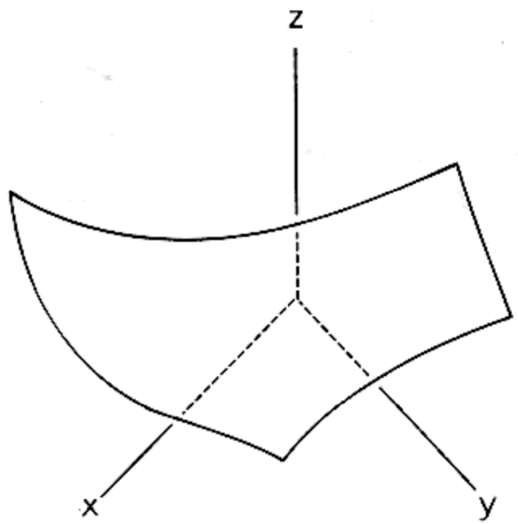
Most “features” of sources of input do not change except over times of more than minutes, e.g. shape (other than configuration), reflectance, texture, temperature, flexibility, sharpness, the pitch of sounds from a source, odors. These features can affect firing. They are nondimensional characteristics.

Curve in 3d



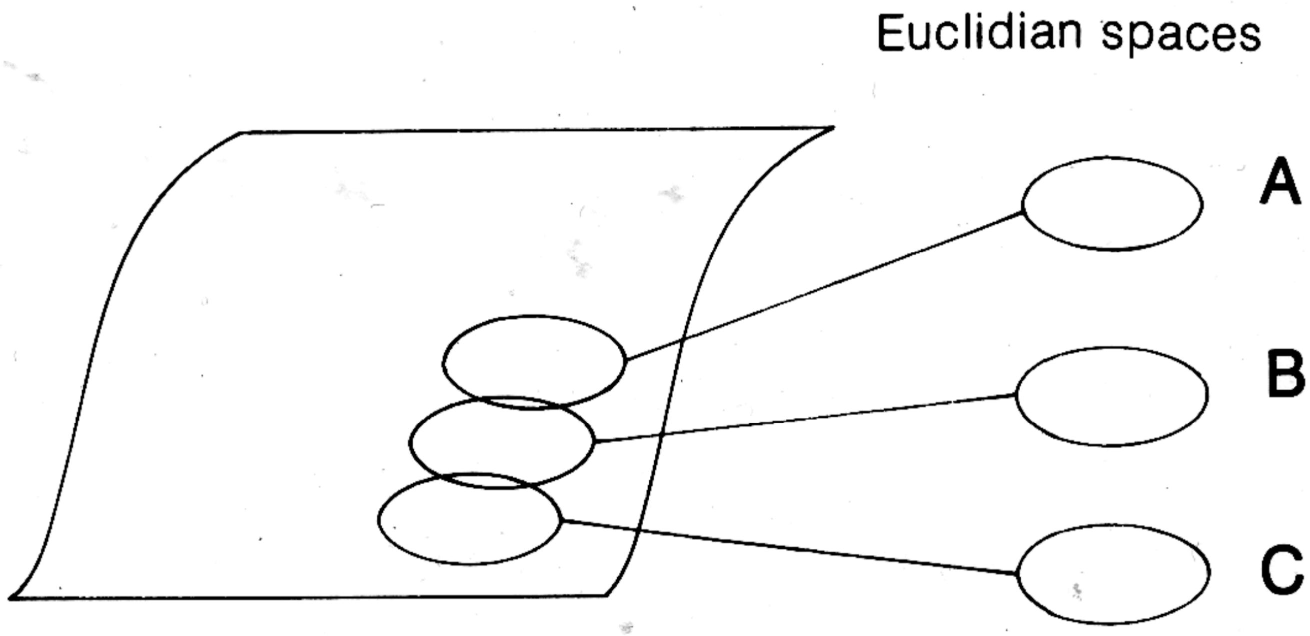
A curve in 3 space.
(a one dimensional manifold embedded in 3 space).
A and B are the ends of the curve.

Surface in 3d

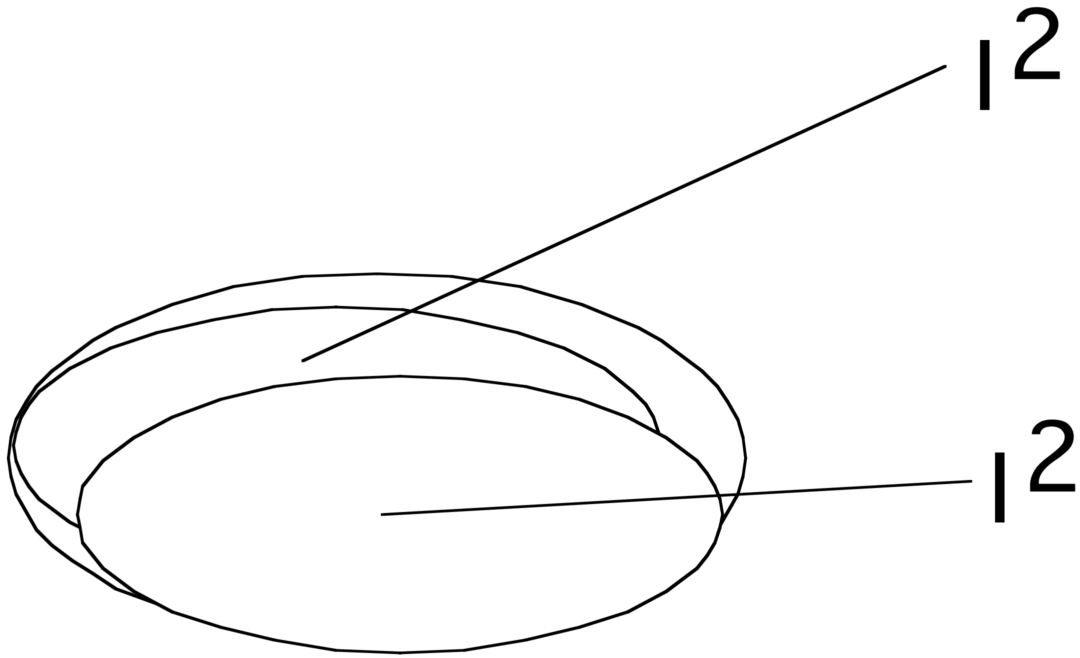


A surface in 3 space:
a two dimensional manifold
embedded in three dimensional
Euclidian space.
This manifold has a boundary.

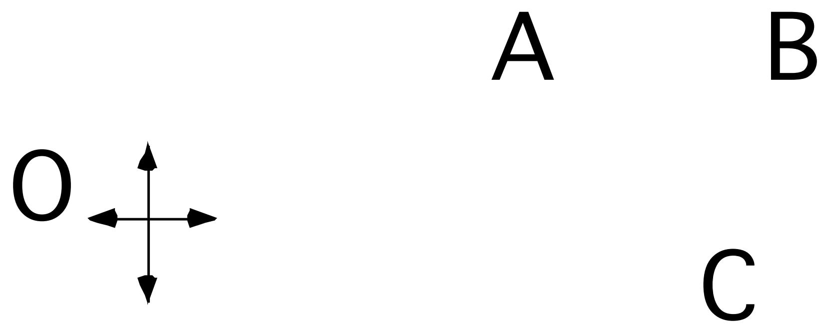
Coordinate charts



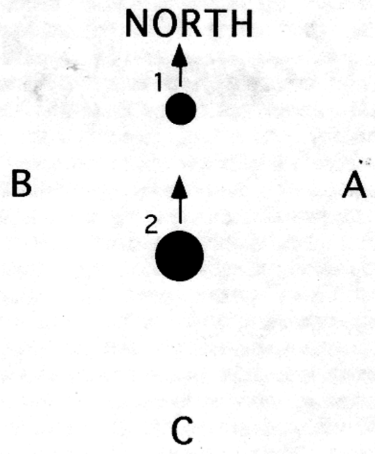
overlapping
coordinate charts



sense organ
and sources

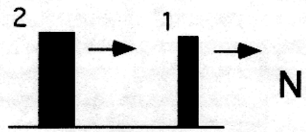


Visual bundle

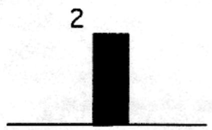


Columns 1 and 2 moving North as seen from above, with observer viewing sites A, B, and C.

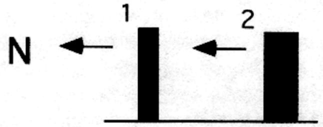
Columns moving North as seen from A



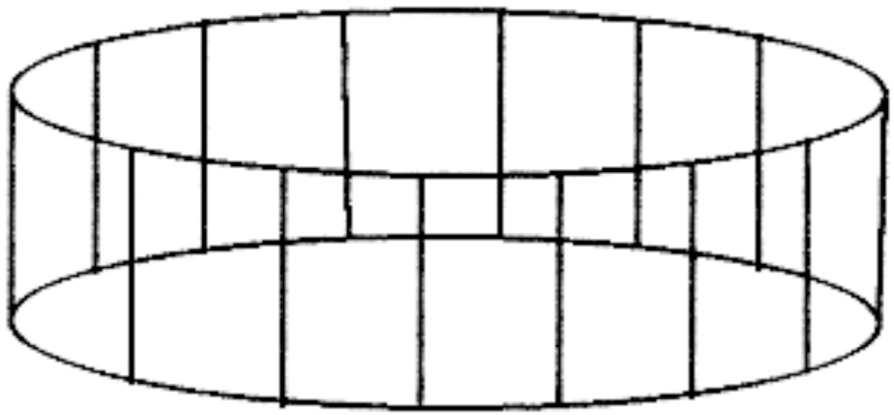
Columns moving North as seen from C



Columns moving North as seen from B

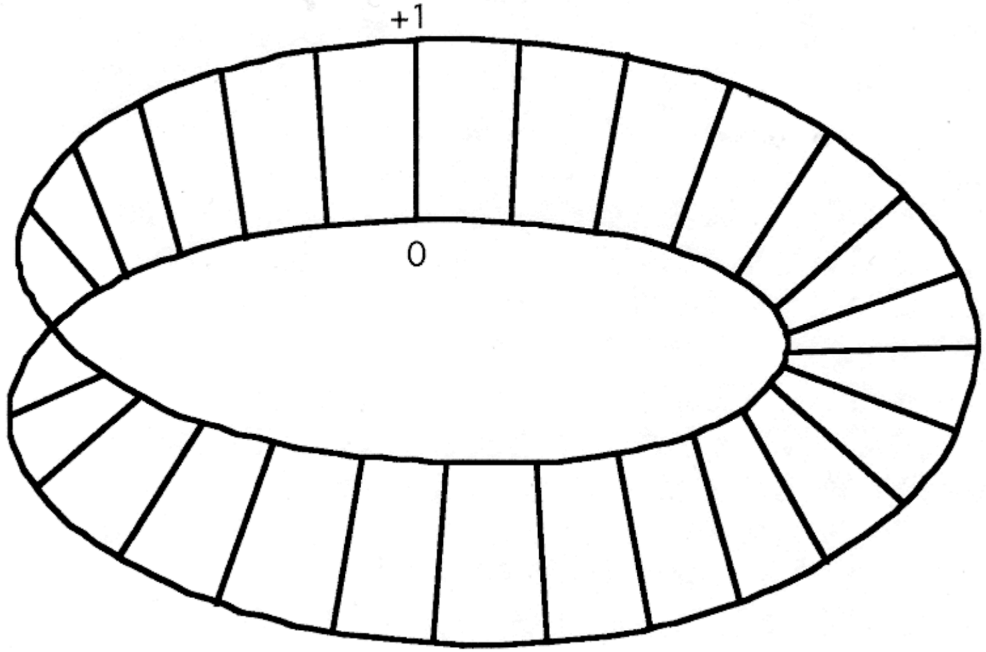


cylinder



CYLINDER

Moebius



MOEBIUS BAND

bundle

Base, B; $b \in B$
Total Space, E; $(b, x) \in E$
Projection from the total space to the base, p
$p: E \rightarrow B$
$p(x, b) = b$

Fiber of the bundle, F; each x is homeomorphic to F

$p^{-1}(b) = (x, b)$, x is the "fiber over the base point b".

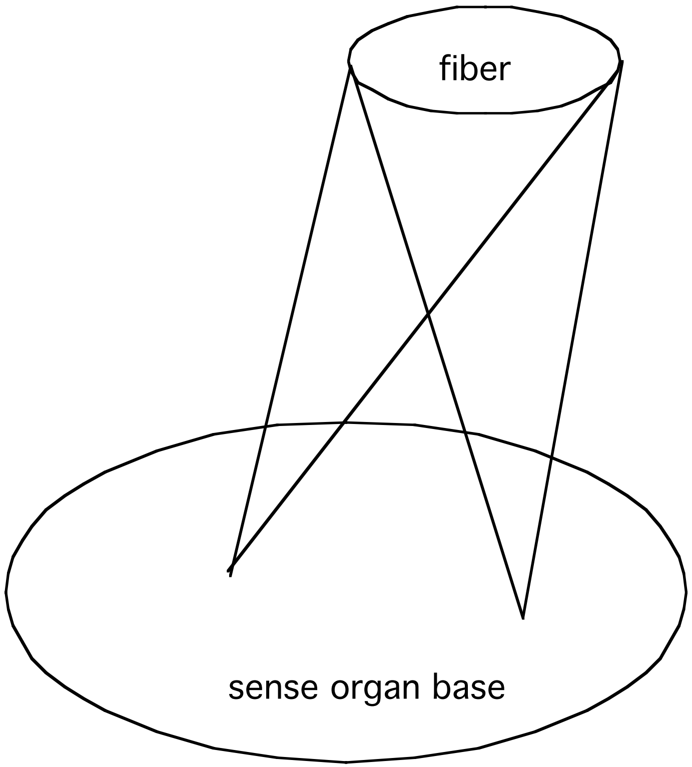
$B \times F$ is a "trivial" bundle

1. Locally a bundle is a product. A product, say $B \times F$, is a bundle but is called a "trivial" bundle.

2. The coordinates of the base are "ordinary" coordinates. The "coordinates"of x depend on its base point. Rules exist for these "coordinates".

3. The relation of the base and the fiber of the bundle are not symetrical. The coordinates of the base do not depend on the fiber of the bundle.

Base and fiber



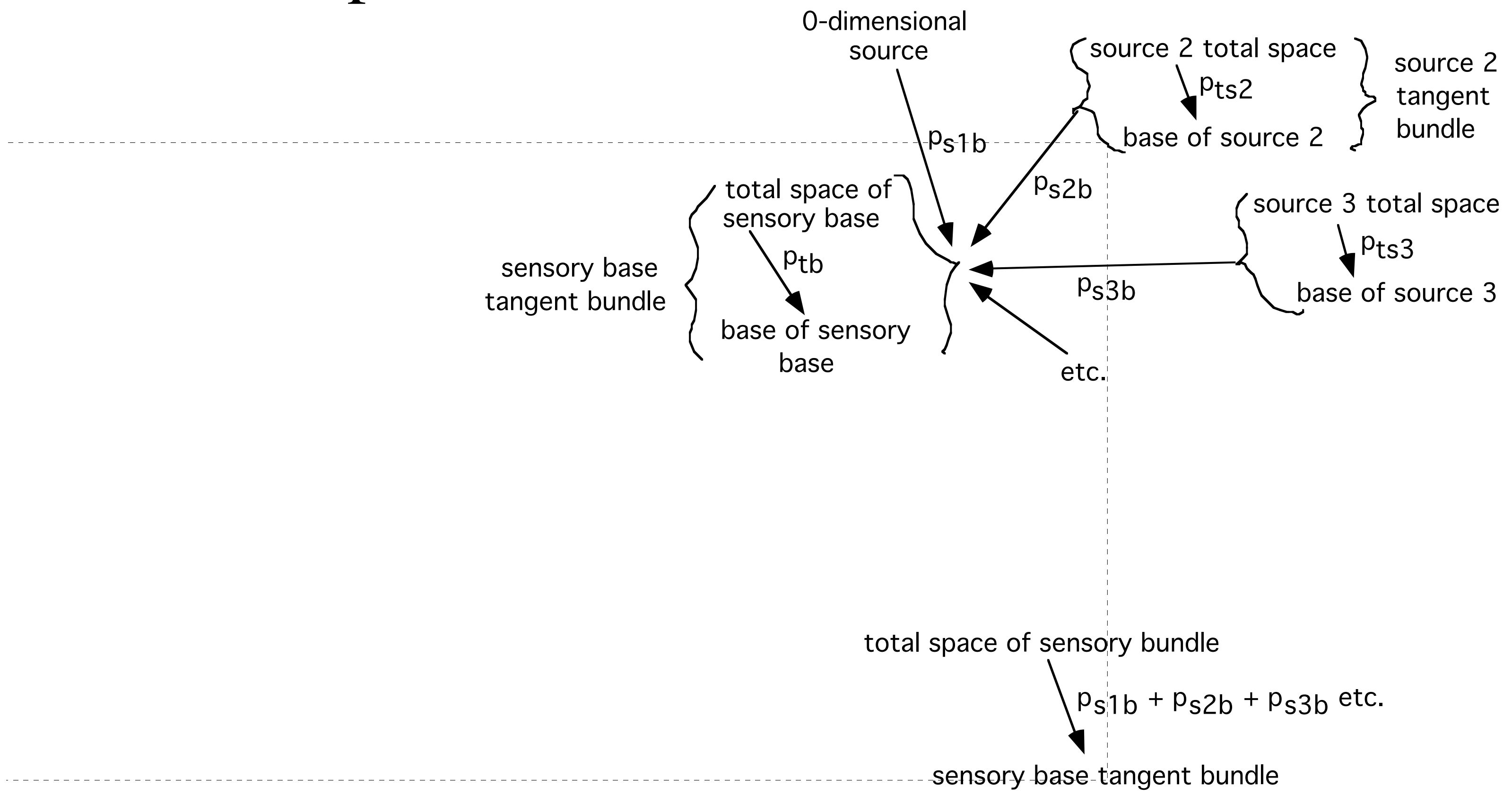
Structure of sensory bundles

The ways the sense organ can move in the external world (the sense organ base) For vision: eye movements, translation and rotation of the head, homeomorphic to $\mathbf{I}^2 \times \mathbf{S}^3 \times \mathbf{I}^3$, plus a tangent space $\mathbf{I}^6$. Each of these components has a projection to a 1- or 2-dimensional manifold.

The ways each source of stimuli can move or stimuli can change in the external world, plus a tangent space.

total space

SENSORY BUNDLE STRUCTURE



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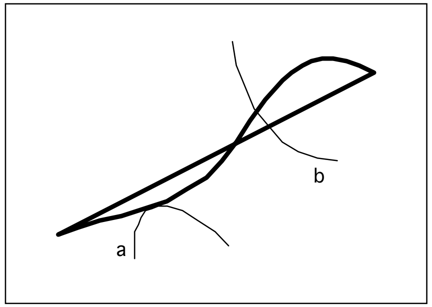
Implications of sensory bundle structure

Location and orientation (yaw, pitch, and roll) of the head are in absolute space. Projections from sensory bundles create the manifolds of location and yaw and its tangent space.

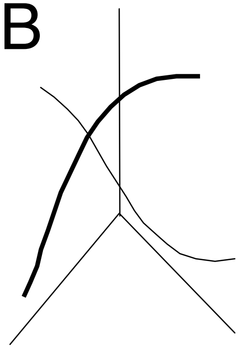
An inverse projection from a point in the sense organ base depends on where the sense organ is, so is not in absolute space. It is in a sense organ/source space, i. e. egocentric space.

transversality

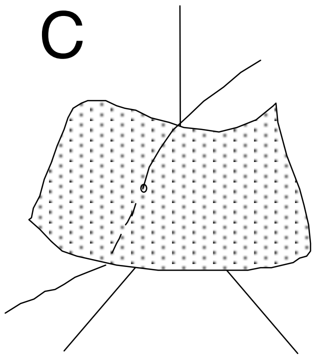
A



B



C



Implications of transversality of the experiential curve in a manifold

1. Points in the experiential curve can happen a second time. In a manifold in which the experiential curve cannot transversally intersect itself, nothing can happen a second time.

2. A Hebbian mechanism often increases the strength of synapses a small amount for each recurrence, so the input and the output must be able to recur: both input and output must be 1- or 2-dimensional. Hebbian mechanisms will strengthen maps producing 1- or 2-dimensional manifolds in which a point of the experiential curve can recur. Manifolds with a dimensionality of 3 or more in which event cannot recur will not have their synaptic strengths increased. Manifolds in which events can recur are selected. Sensory bundles, with a dimensionality of more than 2, have submanifolds with smaller dimensionality (e. g. blobs and interblobs).

3. If learning occurs in one instance of an event, it is useless experiential curve can visit that site a second time. Learning using more than one instance of an event must occur in 1 - or 2-dimensional manifolds in which the embedding is about the same in each occurrence. Recognition and recall requires about the same embedding.

Extrinsic and intrinsic spaces

extrinsic	intrinsic
describes the physical world: changes in neuronal firing rate space and the non-neural world	describes organization of the neural and non-neural physical world
potentially very high dimension	low dimension
metric	may inherit metric from extrinsic space
nothing can recur	many events can recur

Embedding

The embedding of the same intrinsic manifold in extrinsic space (firing rate space) may change with no change in the intrinsic space.

1. motor embedding change (largely due to cerebellum)

2. motivational embedding change (including emotion)

3. change from one coordinate chart to another overlapping coordinate chart

4. embedding of dyadic combination ("matrix multiplication"), action of a group on manifold (ideothetic), tangent space extrapolation, and other metric like functions

5. development and pathology

6. individuation (class vs specific individual animal, object, or environment)

7. variance in embedding

All neurons have a maximum and minimum firing rate so the embedding space has a boundary.

Embedding does not change in perpetual systems.

The inverse projection from a point in the sense organ base is a point if the source is stationary, and can be up to 6-dimensional for each topologically distinct source (mobile source). However, the sense organs move which can contribute sense organ/source dimensions to this inverse projection. In order for the experiential curve to transversally intersect itself, i. e. for an experience to be repeated, the dimensionality of the manifold can be at most 2, and the number of non dimensional characteristics of the non-neural world should not be too large.

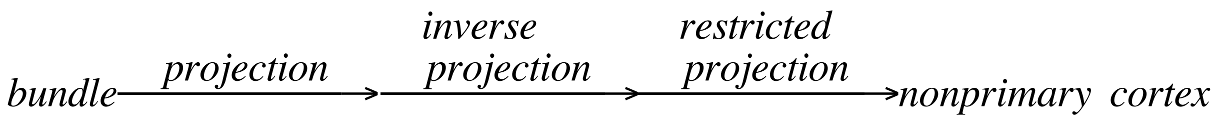
1. The inverse projection from a point in the sense organ base has 6-dimension sense organ-source dimensions for each source (plus tangent dimensions) as either the sense organ or the source moves. 1-d or 2-d manifolds with sense organ/source dimensions can be projections from the inverse projection.

2. Each mobile source introduces up to 6-dimensions (plus tangent dimensions). At most one source is considered at a time, which means the projection from the inverse projection from a point in the sense organ base is only a projection from one source, a restricted projection.

For both the stationary (0-dimensional) and mobile sources in a sense organ/source manifold with many local overlapping coordinate charts, it would often be very difficult or impossible to intersect the experiential curve. Also it is sometimes useful for the brain to intersect the experiential curve using only limited local coordinate charts, whose embedding with variance overlap. Since coordinate charts define a manifold, limiting the local coordinate charts limits the embedding space. For topographically organized manifolds, this often means limiting the manifold to inputs on a limited part of the sense organ surface.

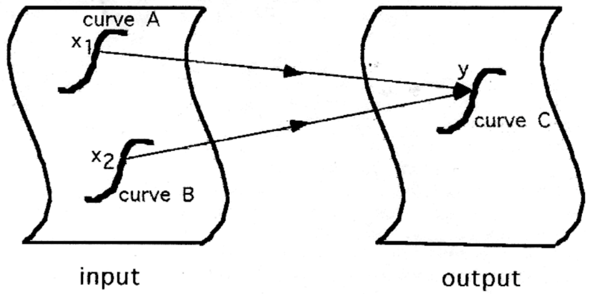
A manifold derived from inverse projection from a point of the sense organ base with limited dimensions, sources, and coordinate charts is a restricted projection from the inverse projection, defining a singled-out-source.

The consequence of this restricted projection is usually called "attention".



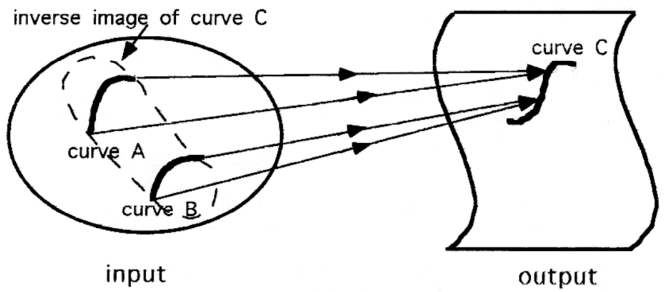
Inverse image topology

Case 1. Dimensions of the input and output are equal



Points in both curves A and B cause points in curve C. Either x_1 or x_2 causes y, but any two points causing the same point must be disjoint. Subsets of curve A and curve B causing the same open sets of curve C are considered the same. Points causing the same result are identified.

Case 2. Dimension of input > dimension of output.



Any point on curve A causes the same point on curve C. Any point on curve B causes the same point on curve C. The input is 2-dimensional and the output is 1-dimensional. The inverse image of curve C is 2-dimensional. When all points causing the same result are indentified under the inverse image topology the input becomes 1-dimensional and the map becomes a homeomorphism.

Inverse image classification and topology

$$A \rightarrow B \rightarrow C$$

Where A is the non-neural world, and B and C are neural classes.

1. Inverse image classification: Define an intrinsic manifold in each neural class, $intM_A$, $intM_B$, and $intM_C$. Let all of inputs from $intM_B$ that cause the same point in $intM_C$ be equivalent. The points of $intM_C$ define a set of equivalence classes in $intM_B$, denoted "$intM_B/\sim$" and called "equivalence classes modulo $intM_C$". The map

f: $intM_B/\sim \rightarrow intM_C$ is 1-to-1 and onto, a bijection.

Let all of the inputs from $intM_A$ that cause the same point in $intM_B/\sim$ be equivalent, defining yet another set of equivalent classes, denoted $intM_A/\sim\sim$ and called "equivalence classes of $intM_A$ modulo $intM_C$". We call $intM_C$ the "generating manifold" for both sets of equivalence classes. The map

g: $intM_A/\sim\sim \rightarrow intM_B/\sim$ is 1-to-1 and onto, a bijection.

2. Composition: The maps f and g can be composed,

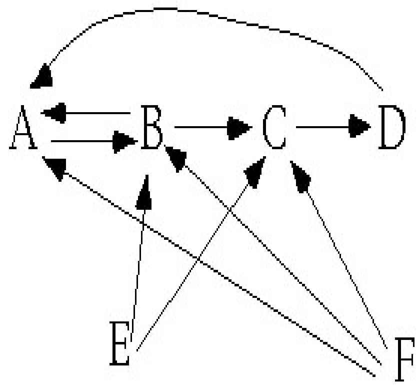
f o g = h: $intM_A/\sim\sim \rightarrow intM_C$, which is 1-to-1 and onto, a bijection.

3. Inverse image topology: Give $intM_C$ any topology. Define an open set in $intM_B/\sim$ (or in $intM_A/\sim\sim$) to be the set that is the image of an open set in $intM_C$ via map f (or h). The topology so defined is called the "inverse image topology in $intM_B/\sim$ (or $intM_C/\sim\sim$) induced by $intM_C$". For this topology $intM_C$ is called the "generator" of the topology or "the generating manifold".

With this topology, maps f and g are homeomorphisms and their composition h is also a homeomorphism. We have a homeomorphism between a generating manifold in a sensory system and a Euclidean description of the non-neural world.

Since a postsynaptic neuron can be fired by many combinations of input that the postsynaptic neuron cannot distinguish, the inverse image classes/topologies describe the non-neural world from the "point of view" of the generating manifold.

Forward classification and topology



where A is a dynamical system and D is the non-neural world.

1. Forward classification: Define an intrinsic manifold for each neural class, $intM_A$, $intM_B$, etc. Let all possible consequences in $intM_B$ of a single curve of $intM_A$, as $intM_E$ and $intM_F$ vary, define a class of curves in $intM_B$. The set of curves of $intM_A$ define classes of curves in $intM_B$, denoted $intM_B/*$ and called "forward classes generated by $intM_A$". (These are not equivalence classes). The map

$f: intM_A \rightarrow intM_B/*$ is 1-to-1 and onto, a bijection.

Let all of the consequences in $intM_C$ of a set of curves in $intM_B/*$ as $intM_E$ and $intM_F$ vary define yet another set of classes of curves of $intM_C$, denoted $intM_C/**$ and called "the forward classes in $intM_C$ generated by $intM_A$". The map

$g: intM_B/* \rightarrow intM_C/**$ is 1-to-1 and onto, a bijection.

similarly define $intM_D/***$ and $h: intM_C/** \rightarrow intM_D/***$

2. Composition: The maps f, g, and h can be composed,

$h \circ g \circ f = k: intM_A \rightarrow intM_D/***$, which is 1-to-1 and on to, a bijection.

3. Forward topology: Give $intM_A$ any topology. Define an open set in $intM_B$/* (or in $intM_C$/** or $intM_D$/***) to be the set that is the image of an open set in $intM_A$ via the map f (or g o f, or k). The topology so defined is called the "forward topology coinduced in $intM_B$/* (or $intM_C$/** or $intM_D$/***) by $intM_A$". For this topology $intM_A$ is called the "generator" of the topology or the "generating manifold".

With this topology, maps f, g, and h are homeomorphisms, and their compositions, f o g, and k are homeomorphisms. We have a homeomorphism between the generating manifold of a dynamical system and a Euclidean description of the non-neural world.

Since a dynamical system cannot know what its consequences will be, even with feedback, these forward classes/topologies describe the non-neural world from the "point of view" of the generating neural class.

Implications of internal classes/topologies

The fact that with inverse image and forward classes/topologies the compositions from generating manifolds to a Euclidean descriptions of the non-neural world are homeomorphisms means:

1. The compositions are homeomorphisms to Euclidean space, and therefore are coordinate charts of a manifold. The inverse image and forward classes/topologies not only describe sensory systems and dynamical systems and their consequences, but they also have the same structure.

2. The inverse image classification/topologizing moves in the opposite direction from neural causation. The forward classification/topologizing move in the same direction as neural causation.

3. All of the intermediate steps between a Euclidean description of the non-neural world and the generating manifold become invisible to the generating manifold with the composition.

space × class/topology table

	extrinsic space	intrinsic space
usual topology	cellular and molecular mechanisms	behavioral descriptions and analysis
internal classes/topologies	psychopharmacology neuropsychology	mind

Summary

1. The firing of most classes of neurons can be described as a low dimensional manifold embedded in a much higher dimensional firing rate space, an intrinsic space embedded in a much higher dimensional space, the extrinsic space.
2. Sensory systems have a bundle structure. The visual, auditory, and cutaneous systems are nontrivial bundles. The sense organ base of location and orientation describes absolute space.
3. The 1-dimensional experiential curve can only transversally intersect itself, i. e. recur, if it is in a 1- or 2-dimensional manifold.
4. The embedding of an intrinsic space can change without affecting the intrinsic manifold, but with other consequences.
5. Items #2, 3, and 4 combine to define attention.
5. The usual topology can used on all maps between intrinsic or extrinsic manifolds, the “point of view” of a scientist.

From the “point of view” of classes of neurons in a sensory system an inverse image classification/topology is used.

From the “point of view” of a dynamical system a forward classification/topology is used.

6. We can look at brain activity and its consequences (behavior and consciousness) in four ways depending on whether considered as extrinsic or intrinsic space or the usual topology or inverse image or forward classes/topologies.