Dynamical systems: an elementary introduction

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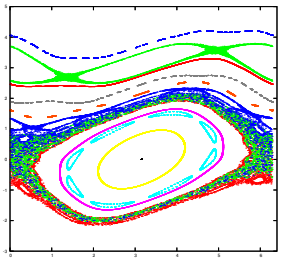

The present text is the enlarged version of the preparatory course which was taught on June 9, as an introduction to the dynamical system mini-course of the summer school.

We present some basic definitions first on discrete dynamical systems, then on systems of ordinary differential equations, and finally on the relations between them. In order to make the text as self-contained as possible, an appendix on calculus has been added.

1 Discrete dynamical systems

1.1 From a mathematical point of view, the definition of a discrete dynamical system is incredibly simple: it is just a map from any set in itself.

The set is called **the phase space**; it represents all the possible states of the system under study. We shall denote it by M, and we give ourselves a map

$$F \,:\, M \to M,$$

which is supposed to describe the evolution of the system between any two instants of time: it sends a point p of M (a state of the system at a given instant) onto a new point $F(p)$ (the new state of the system, at the next instant).

We are interested in the long-term evolution of the system. Thus, if the initial state of the system is known, we need to apply the map F to get the next state, and to apply it again to know what happens after this, and again, *i.e.* to **iterate the map** F.

What is discrete here is the time:[1] measuring the state of the system at time $t = 0$, then $t = 1$, and then $t = 2 \ldots$ (with a given unit of time), we get a sequence of results $p_0, p_1, p_2, \ldots$ satisfying the induction formula

$$p_{n+1} = F(p_n).$$

[1] *Discrete* as opposed to *continuous*—see Section 2.

We are thus modelling a deterministic process: as soon as the initial state p_0 is known, the future evolution of the system is determined, namely $p_1 = F(p_0)$ at time 1, $p_2 = F(p_1) = F(F(p_0))$ at time 2, $p_3 = F(p_2) = F(F(F(p_0)))$ ...; only the initial point p_0 is arbitrary.

1.2 A simple example is the growth of a population with a fixed fertility rate $\lambda > 1$: the state of the system is completely described by the number p_n of individuals at generation number n, which obeys the law

$$p_{n+1} = \lambda p_n$$

(which amounts to saying that the number of individuals at generation $n+1$ is λ times the number at generation n). Another example is the disintegration law of a radioactive substance: assuming that a proportion μ of nuclei is transformed between two measurements, we find that the quantity p of radioactive substance obeys the law

$$p_{n+1} = p_n - \mu p_n.$$

In both cases, one can take $M = \mathbb{R}^+$, the set of non-negative real numbers, and $F(p) = \lambda p$, with $\lambda = 1 - \mu \in]0, 1[$ in the second example.

In both examples, starting with an arbitrary $p_0 > 0$, we find the exponential law[2]

$$p_n = \lambda^n p_0,$$

which describes unbounded proliferation when $\lambda > 1$ (then p_n tends to infinity as $n \to \infty$), or extinction when $0 < \lambda < 1$ (then $p_n \to 0$).

1.3 But these examples are terribly simple. There is in general no closed formula relating the state p_n at time n and the initial state p_0, even if the map F does not look too complicated.

Figure 1 shows[3] what happens with the so-called "standard map" (a famous model of mathematical physics, but we shall not discuss what this model is for): here, two variables x and y are needed to describe a point p of the phase space, and the map is

$$F(x, y) = (x', y'), \quad \left| \begin{array}{l} x' = x + y + \varepsilon \sin x \\ y' = y + \varepsilon \sin x \end{array} \right. \tag{1}$$

(the positive number ε is a fixed parameter).

Given an initial state (x_0, y_0), we can compute $x_1 = x_0 + y_0 + \varepsilon \sin x_0$, $y_1 = y_0 + \varepsilon \sin x_0$, and then $x_2 = x_1 + y_1 + \varepsilon \sin x_1$, $y_2 = y_1 + \varepsilon \sin x_1 = y_0 + \varepsilon \sin x_0 + \varepsilon \sin(x_0 + y_0 + \varepsilon \sin x_0)$, but the formulas get more and more complicated pretty quickly, and it is for large n and arbitrary (x_0, y_0) that we'd like to know the behaviour of (x_n, y_n)! Figure 1 shows, in different colors, 11 initial conditions and

[2]The sequence $\{\lambda^n p_0\}$ is a geometric progression with ratio λ, but we call this an exponential law because $\lambda^n = \mathrm{e}^{an}$ with $a = \log \lambda$.

[3]Warm thanks to Ph. Robutel, who kindly accepted to make this picture for this course!

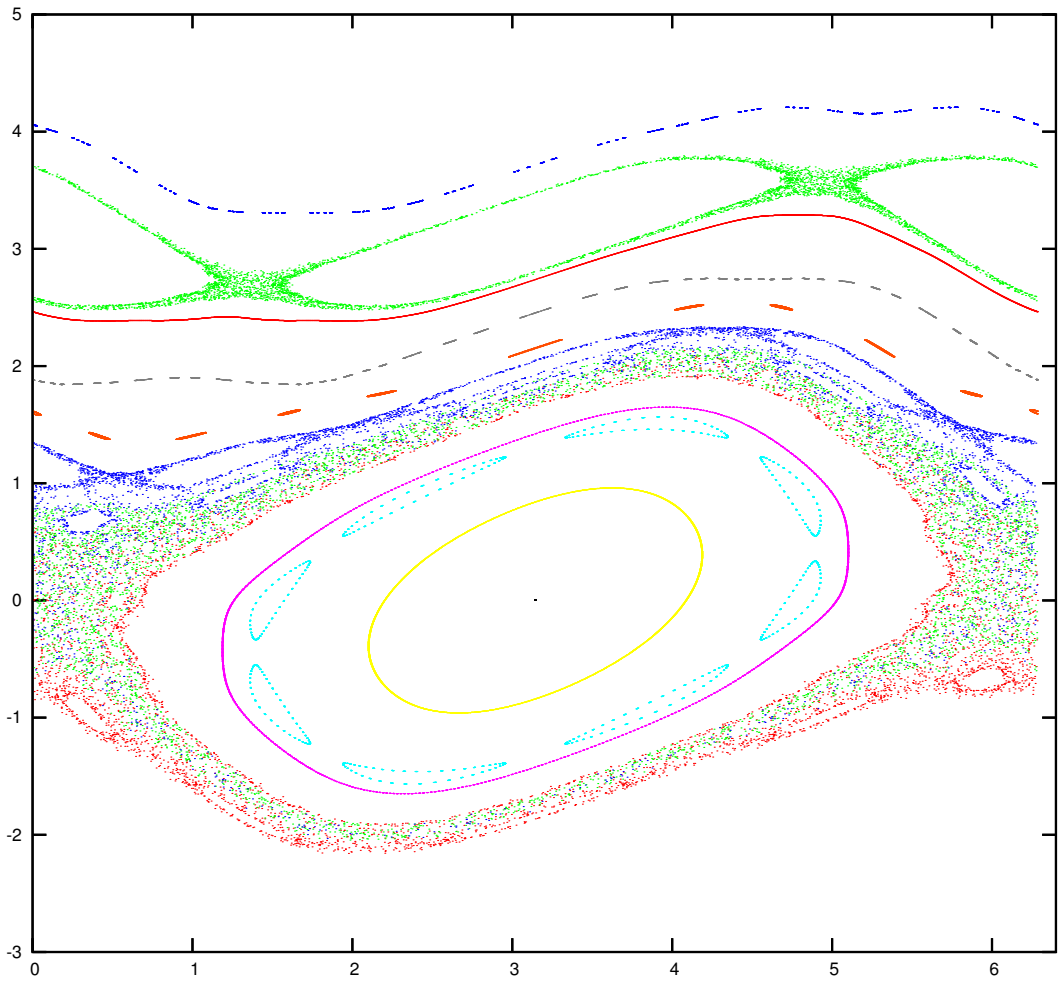

Figure 1: 11 orbits of the standard map, with $\varepsilon = 0.9$.

the next 5000 points they give rise to: the behaviour of this systems indeed seems quite complicated...

1.4 In the general abstract situation $F : M \to M$, any point p of M can be considered as an **initial condition** $p = p_0$, corresponding to the state of the system at the origin of time. The future evolution of the system is then determined as a sequence $\{p_0, p_1, p_2, \ldots\} = \{p_n\}_{n\in\mathbb{N}}$, which is called the **forward orbit of** p. At time $n \geq 1$,

$$p_n = \underbrace{F(F(\cdots F}_{n\,\text{times}}(p_0)\cdots)) = F^n(p_0).$$

This is the definition of a map $F^n : M \to M$, which is the n^{th} **iteration of** F and which describes the evolutions of the systems between two instants separated by n units of time; we also define $F^0 = \mathrm{Id}$, the identity map of M (*i.e.* $F^0(p) = p$: if the initial condition is p, then the state at time 0 is still p!).

The maps we shall be interested in will be **invertible**: there exists a so-called reciprocal map $G : M \to M$ such that $F(G(p)) = p$ for any p. In other words, for any initial condition $p_0 = p$, the past evolution is also uniquely determined: we had $p_{-1} = G(p)$ at time -1, and previously $p_{-2} = G(G(p))\ldots$ It is convenient to set $F^{-1} = G$, and more generally $F^{-n} = G^n$. The bi-infinite sequence $\{p_n = F^n(p)\}_{n\in\mathbb{Z}}$ is then called the **complete orbit of** p.

The (obvious) property $F^n(F^\ell(p_0)) = F^{\ell+n}(p_0)$ can be written

$$F^n \circ F^\ell = F^{\ell+n}$$

(using the symbol $\circ$ for functional composition). It amounts to saying that, when we take an arbitray point $F^\ell(p_0)$ on the orbit of p_0 as new initial condition we get the same orbit, except for the time-shift $n \mapsto \ell + n$. Each orbit thus represents a possible history of the system, on which the choice of the origin of time is arbitrary, and the phase space can be regarded as the disjoint union of the orbits (each point of the space belongs to one orbit, namely to its own orbit, but to no other). The theory aims at understanding all the possible behaviours of the system, *i.e.* the various types of orbits and their arrangement one with respect to the other.

An orbit may be reduced to a single point: this happens if and only if $F(p_0) = p_0$, the point p_0 is then called a **fixed point** of F (or an equilibrium: if the system is in this state at some instant, in fact it stays in this state forever). For instance, the point $(0, 0)$ is a fixed point of the aforementioned standard map—can you find another one? (remember that the sin function vanishes at 0 and π.)

An orbit may be reduced to a finite sequence of points $\{p_0, p_1, \ldots, p_{N-1}\}$, *i.e.* it may happen that p_N and the initial condition p_0 coincide, but then p_{N+1} and p_1 must also coincide, etc. (because the system is deterministic: whenever you are at a point p_0, you must be at $p_1 = F(p_0)$ at the next instant). This happens if and only if p_0 is a fixed point of F^N; its orbit is then called N**-periodic** (the system passes through the states $p_0, p_1, \ldots, p_{N-1}$, and then indefinitely repeats this finite sequence of states in the same order: $F^{\ell+kN}(p_0) = F^\ell(p_0)$ for all $\ell \in \{0, \ldots, N-1\}$ and $k \in \mathbb{Z}$).

On figure 1, we can guess that the point $(0, \pi)$ has a 2-periodic orbit.[4] Looking at the picture, can you guess in what region of the phase space we can find a 4-periodic orbit, a 8-periodic orbit, or a 10-periodic orbit?

1.5 The complexity of the system depends on the number of variables needed to describe a state, *i.e.* on the dimension of the phase space. The dimension was 1 in our first examples in § 1.2, and it was 2 for the standard map of § 1.3, which was already a pretty complicated system. But to model a real system one sometimes needs much more dimensions, which makes things harder to analyse and to visualize...

The complexity depends also on the shape of the map F. The simple examples of § 1.2 can be generalized to an arbitrary number of dimensions by what is called a **linear system**. Here, we let $d \geq 1$, and the phase space is $\mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$, *i.e.* a point p is determined by d real numbers (its coordinates), which we find convenient to write in a column

$$p = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}.$$

We consider a map F of the form

$$F(p) = \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_d \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_d \\ \vdots \\ a_{d,1}x_1 + a_{d,2}x_2 + \cdots + a_{d,d}x_d \end{pmatrix},$$

where the coefficients $a_{i,j}$ are fixed real numbers. The system is thus completely determined by these numbers, that we find convenient to write in a matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,d} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,d} \\ \vdots & & \ddots & \\ a_{d,1} & a_{d,2} & \cdots & a_{d,d} \end{pmatrix}.$$

This is the definition of the linear map from $\mathbb{R}^d$ into itself associated with the matrix A; one usually writes $F(p) = A \cdot p$.

[4] In fact, $(x_0, y_0) = (0, \pi)$ implies that $(x_n, y_n) = (n\pi, \pi)$, which is not a periodic orbit in $\mathbb{R} \times \mathbb{R}$, but it is natural here to consider the first variable as an angle, *i.e.* to identify any two points the first coordinate of which differ by an integer multiple of 2π (because the system depends on x only through $\sin x$, which is a 2π-periodic function of x). With this convention $(x_2, y_2) = (2\pi, \pi)$ and (x_0, y_0) are identified.

The whole figure 1 was made with that convention: each time a point (x_n, y_n) was computed, an appropriate multiple of 2π was substracted from the first coordinate in order to yield a point in $[0, 2\pi[\times \mathbb{R}$. We should consider as the true phase space of this system the set $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the circle (with angular coordinate x).

Linear algebra helps us to iterate such a map in a very efficient way, by simply manipulating the matrix A. For instance, it tells us how to recognize whether the map is invertible (as defined in § 1.4) by computing a certain combination of the coefficients $a_{i,j}$ (the so-called determinant of the matrix). It tells us that iterating n times the map F amounts to multiplying n times the matrix A by itself, where multiplication of matrices is defined by certain rules, so that F^n is itself a linear map: $F^n(p) = A^n \cdot p$, where A^n is a certain matrix.

But it also tells us how one can compute the matrices A^n for all values of n. Quite often it is indeed possible to **diagonalize** the matrix A, *i.e.* to reduce the situation to the simplest possible form:

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_d \end{pmatrix}, \quad F : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_d x_d \end{pmatrix}, \tag{2}$$

in which case we simply find d independent exponential laws

$$A^n = \begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_d^n \end{pmatrix}, \quad F^n : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1^n x_1 \\ \lambda_2^n x_2 \\ \vdots \\ \lambda_d^n x_d \end{pmatrix}.$$

The complete analysis of such a system is then easily achieved.

The reduction of the system to the diagonal form (2) is done by a linear change of coordinates, which may be not so easy to find, but this task needs to be performed only once and then all the orbits are easily determined!

1.6 Here is a simple example with 2 dimensions: suppose the coordinates of p and those of $p' = F(p)$ are related by the linear formulas

$$x_1' = -3x_1 + 7x_2, \quad x_2' = -\frac{7}{2}x_1 + \frac{15}{2}x_2.$$

A little practice in linear algebra will lead you to define new coordinates

$$\begin{aligned} X_1 &= -x_1 + 2x_2 \\ X_2 &= x_1 - x_2, \end{aligned}$$

from which the old ones can be recovered by the formulas

$$\begin{aligned} x_1 &= X_1 + 2X_2 \\ x_2 &= X_1 + X_2. \end{aligned}$$

If $p = \binom{x_1}{x_2}$ has new coordinates $\binom{X_1}{X_2}$, then $F(p) = \binom{x_1'}{x_2'}$ has new coordinates

$$\left| \begin{aligned} X_1' &= -x_1' + 2x_2' = -4x_1 + 8x_2 = 4X_1 \\ X_2' &= x_1' - x_2' = \frac{1}{2}x_1 - \frac{1}{2}x_2 = \frac{X_2}{2}. \end{aligned} \right. \tag{3}$$

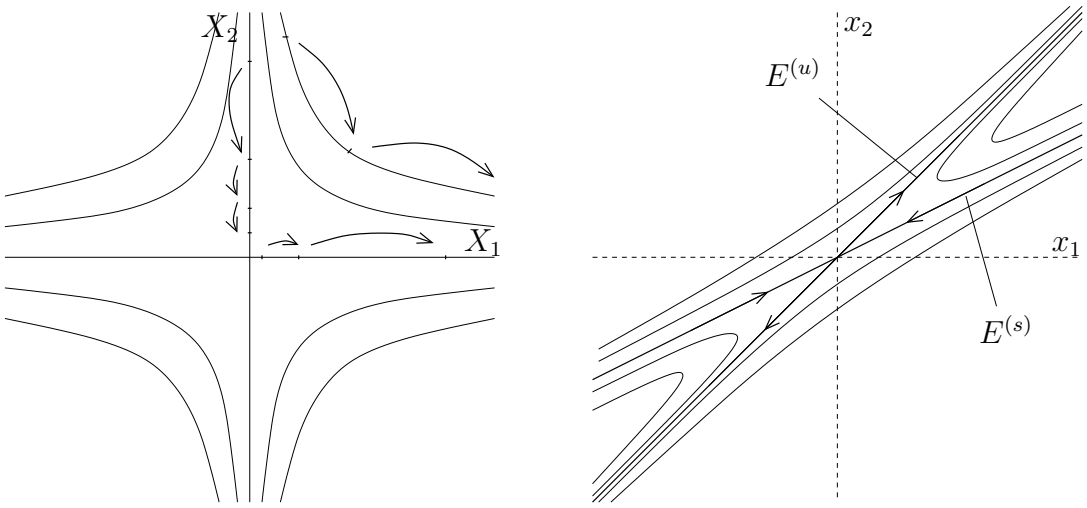

Figure 2: A linear system in 2 dimensions.

In other words, the matrix $A = \left(\begin{smallmatrix} -3 & 7 \\ -\frac{7}{2} & \frac{15}{2} \end{smallmatrix}\right)$ is brought to the diagonal form $D = \left(\begin{smallmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix}\right)$ by the change of coordinates $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix}\right)$. We can then describe all the orbits of the system:

– There is a fixed point at the origin (as always with linear systems).

– Two special families of orbits correspond to the vanishing of one coordinate

$$D^n \cdot \begin{pmatrix} 0 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2^{-n} X_2 \end{pmatrix}, \quad D^n \cdot \begin{pmatrix} X_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4^n X_1 \\ 0 \end{pmatrix}.$$

This means that if the system starts on the axis $\{X_1 = 0\}$, then it stays on it forever: the corresponding straight line in the original coordinates $E^{(s)} = \{-x_1 + 2x_2 = 0\}$ is said to be **invariant by** F (and similarly for the axis $\{X_2 = 0\}$, which corresponds to the straight line $E^{(u)} = \{x_1 - x_2 = 0\}$). Observe that the forward orbit of a point on $\{X_1 = 0\}$ tends to the origin (simply because 2^{-n} tends to 0 as $n \to +\infty$), whereas the forward orbit of a point on $\{X_2 = 0\}$ escapes to infinity; however, the second kind of point is characterized by the fact that the backward orbit tends to the origin (because 4^n tends to infinity as $n \to +\infty$, but tends to 0 as $n \to -\infty$). One can say that the point of $\{X_1 = 0\}$ are **attracted** by the origin and that those of $\{X_2 = 0\}$ are **repelled** by the origin.[5]

– A general initial condition with $X_1 \neq 0$ and $X_2 \neq 0$ does not give rise to an orbit contained in a fixed straight line, but formula (3) shows that $X_1'(X_2')^2 = X_1(X_2)^2$, thus these orbits are contained in the curves $\{X_1(X_2)^2 = \text{constant}\}$, which correspond to hyperbola-like curves in the original coordinates—see Figure 2. In other words, the function φ which associates the value $X_1(X_2)^2$ with any point of the phase space satisfies

$$F \circ \varphi = F,$$

[5]This is a linear instance of the so-called stable or unstable manifolds to be encountered in Section 4.

and is called a **conserved quantity** for that reason; its level curves $\{\varphi(p) = \text{constant}\}$ are thus invariant by F.

1.7 The numbers $\lambda_1, \ldots, \lambda_d$ in (2) are called **eigenvalues**. As we saw in the previous 2-dimensional example, eigenvalues with absolute value smaller than 1 are related to contraction and attractive behaviour along the corresponding directions, whereas eigenvalues with absolute value larger than 1 are related to dilatation and repulsive behaviour.

It may happen that diagonalisation is not possible with real eigenvalues but the system admits complex eigenvalues. The simplest case is

$$A = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix},$$

for which $\lambda_1 = \mathrm{e}^{\mathrm{i}\omega}$, $\lambda_2 = \mathrm{e}^{-\mathrm{i}\omega}$. The associated linear map is a rotation of angle ω in the plane; the n^{th} iterate is then a rotation of angle $n\omega$:

$$A^n = \begin{pmatrix} \cos(n\omega) & -\sin(n\omega) \\ \sin(n\omega) & \cos(n\omega) \end{pmatrix}.$$

As a rule, non-real complex eigenvalues come in pairs of conjugate complex numbers and correspond to oscillatory behaviour. Rotation can be combined with contraction or dilatation; the reader may try to analyse the orbits of the linear map of matrix

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2}\cos\omega & -\frac{1}{2}\sin\omega & 0 & 0 \\ \frac{1}{2}\sin\omega & \frac{1}{2}\cos\omega & 0 & 0 \\ 0 & 0 & 2\cos\omega & -2\sin\omega \\ 0 & 0 & 2\sin\omega & 2\cos\omega \end{pmatrix}.$$

Sometimes, diagonalisation is not possible at all, even with complex numbers, but linear algebra provides us with enough tools to deal with any linear system in any dimension, in a fashion which is almost as satisfactory as in our simple examples.

Non-linear systems are usually much more difficult to understand. Still, one can sometimes find invariant curves, or invariant higher-dimensional invariant surfaces, or special orbits which are attracted or repelled by a fixed point or by a periodic orbit, etc., which help us to understand how the orbit are arranged in the phase space.

2 Differential equations and vector fields

Differential equations give rise to dynamical systems with continuous time, but this requires a more elaborate mathematical apparatus, since we need to rely on concepts like differentiability and derivatives. The reader is referred to the appendix for a brief reminder concerning the notions of open subsets, continuous and continuously differentiable functions, parametrised curves, etc.

2.1 We shall deal with **systems of autonomous differential equations of order** 1, *i.e.* systems of equations of the form

$$\left|\begin{array}{l} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2, \ldots, x_N) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2, \ldots, x_N) \\ \qquad \vdots \\ \dfrac{\mathrm{d}x_N}{\mathrm{d}t} = f_N(x_1, x_2, \ldots, x_N). \end{array}\right. \tag{4}$$

Here $N \geq 1$ is an integer (the dimension) and $f_1, f_2, \ldots, f_N$ are functions. Each f_i is a real-valued function of N real variables, defined on $\mathbb{R}^N$ or maybe on a subset Ω of $\mathbb{R}^N$, which is then supposed to be open. We shall always assume that the functions $f_1, \ldots, f_N$ are continuously differentiable.[6]

The open subset Ω of $\mathbb{R}^N$ on which the right-hand side of the system (4) is defined is called the **phase space**. Like in the discrete case (§ 1.1), it represents all the possible states of the system.

We have a **solution** of the system (4) whenever we have N differentiable functions $x_1, \ldots, x_N : I \to \Omega$, where I is an open interval of $\mathbb{R}$, such that

$$\left|\begin{array}{l} \dfrac{\mathrm{d}x_1}{\mathrm{d}t}(t) = f_1\big(x_1(t), x_2(t), \ldots, x_N(t)\big) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t}(t) = f_2\big(x_1(t), x_2(t), \ldots, x_N(t)\big) \\ \qquad \vdots \\ \dfrac{\mathrm{d}x_N}{\mathrm{d}t}(t) = f_N\big(x_1(t), x_2(t), \ldots, x_N(t)\big). \end{array}\right.$$

for all $t \in I$. In other words, in place of $x_1, \ldots, x_N$ (which are independent variables from the point of view of the functions f_i), one must substitute unknown functions $x_1, \ldots, x_N$ of the independent variable t. This variable t is called **time** because this is what it is supposed to represent.

It is convenient to group together the functions in the right-hand side and to consider the vector-valued function

$$f = (f_1, \ldots, f_N) : \Omega \to \mathbb{R}^N,$$

which is called the **vector field** associated with (4). We also group together the components of a solution: a solution is then viewed as a parametrised curve[7] $\gamma : I \to \Omega$, $t \mapsto \gamma(t) = \big(x_1(t), \ldots, x_N(t)\big)$, such that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = f\big(\gamma(t)\big) \tag{5}$$

[6] See the definition in § 5.3.

[7] See the definition in § 5.7.

for all $t \in I$.

"Vector field" simply means that at each point p of the phase space we are given a vector $f(p)$. We may think of $\gamma(t)$ as of a mobile point which is asked to follow the "infinitesimal trend" which is indicated by the vector field at each instant of time: equation (5) means that the velocity vector[8] at time t must equal the given vector at the location $\gamma(t)$ reached by the mobile point.

We are thus describing the time-evolution of the system by a solution $\gamma = (x_1, \ldots, x_N)$ of (4), which is equivalent to (5). The state of the system at time t is represented by a point $\gamma(t)$ in the phase space, which moves continuously according to the law (5) as time varies. Such a system is deterministic: we shall see in § 2.5 that, as in the discrete case, the evolution of the system (past and future) is completely determined as soon as the state is known at a given instant of time.

2.2 Here is an example discussed in [HW], Part 2, pp. 19–20. In order to model the fluctuations in the populations of sardines and sharks, the Italian mathematician Volterra proposed the following system of differential equations

$$\left| \begin{array}{l} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \quad ax_1 - cx_1x_2 \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -\, bx_2 + dx_1x_2, \end{array} \right. \tag{6}$$

where a, b, c, d are positive parameters.

Here x_1 represents the number of sardines and x_2 the number of sharks,[9] and these two positive numbers completely describe the state of the system (thus $\Omega =]0, +\infty[\times]0, +\infty[$). The idea is that in the absence of sharks, the sardines would tend to proliferate: the rate of increase of their population $\dfrac{\mathrm{d}x_1}{\mathrm{d}t}$ would be proportional to their number x_1, hence the term ax_1; symmetrically, in the absence of sardines, the sharks would not survive and it is assumed in this model that the rate of decrease of their population would be proportional to their number x_2, hence the term $-bx_2$. But there is an interaction between the two populations (best appreciated by sharks), which tends to decrease x_1 and to increase x_2, hence the terms $-cx_1x_2$ and dx_1x_2 (proportional to the total number x_1x_2 of possible encounters).

It turns out that all the solutions of (6) are periodic functions of time (but they have not all the same period), which fits satisfactorily with the periodic fluctuations observed when fishing.

2.3 A **periodic solution** of (5) is a solution $t \mapsto \gamma(t)$, defined for all real values of t, for which there exists a positive number $T > 0$ such that

$$\gamma(t + T) = \gamma(t)$$

[8] See the definition in § 5.7.

[9] Of course, in this mathematical model, we forget that the numbers of individuals should be integer numbers and treat them as real numbers which change continuously, an approximation which is tolerable when these numbers are large enough. In fact an integer-valued function cannot be continuous (nor differentiable), unless it is constant.

(compare with the definition of periodic orbits of discrete systems in § 1.4).

An **equilibrium point** of (5) is a constant solution, *i.e.* a point p_0 of the phase space such that $t \mapsto \gamma(t) = p_0$ is a solution. Since we then have $\gamma'(t) = 0$ for all t, this is equivalent to having $f(p_0) = 0$: the vector field must vanish at p_0.

Observe that Volterra's prey-predator system (6) has exactly one equilibrium point: $(\frac{b}{d}, \frac{a}{c})$. Maybe the reader wonders how we know that all the other solutions of this system are periodic; this is related to the existence of a first integral.

A **first integral** of the system (5) is a differentiable function on the phase space, $G : \Omega \to \mathbb{R}$, such that

$$\mathrm{d}G(p) \cdot f(p) = 0, \qquad p \in \Omega \tag{7}$$

(the left-hand side is the differential of G at p, which is a linear map from $\mathbb{R}^N$ to $\mathbb{R}$, evaluated on the vector $f(p) \in \mathbb{R}^N$; it is equivalent to say that the function $\dfrac{\partial G}{\partial x_1} f_1 + \cdots \dfrac{\partial G}{\partial x_N} f_N$ vanishes identically—see § 5.2). In view of (5) and according to formula (21) in the appendix, this is equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}\big[G\big(\gamma(t)\big)\big]_{|t=t_0} = 0$ for any solution γ at any t_0, *i.e.* to the fact that $t \mapsto G\big(\gamma(t)\big)$ is a constant function for any solution γ, hence:

> *A first integral is a function which takes the same value along any solution of the system.*

(Compare with the definition of a conserved quantity for a discrete system in § 1.6.)

Level sets of first integrals are examples of invariant sets. A subset Σ of the phase space is said to be **invariant** for (5) if, whenever a solution has a point in Σ, all this solution is contained in Σ (more precisely: for any solution $\gamma : I \to \Omega$, the existence of a $t_0 \in I$ such that $\gamma(t_0) \in \Sigma$ implies that $\gamma(t) \in \Sigma$ for all $t \in I$).[10] Thus, G is a first integral if and only if, for every c in the range of G, the level set $\Sigma_c = \{\, p \in \Omega \mid G(p) = c \,\}$ is invariant.

In the case of system (6), one can check that the formula

$$G(x_1, x_2) = x_1^b x_2^a \, \mathrm{e}^{-dx_1 - cx_2}$$

defines a first integral (§§ 5.3–5.4 and § 5.6 contain enough information to lead you to $\frac{\partial G}{\partial x_1}(x_1, x_2) = x_1^{b-1} x_2^a (b - dx_1) \, \mathrm{e}^{-dx_1 - cx_2}$, and a similar formula for $\frac{\partial G}{\partial x_2}(x_1, x_2)$, from which (7) follows). This is sufficient to draw all solutions of (6), since the level sets $\{G = \text{constant}\}$ are curves in Ω, and they must be invariant. It turns out that these level sets are closed curves; for each solution, the corresponding mobile point is constrained to travel periodically along the curve in which the solution is contained—see Figure 3.

When dealing with systems of dimension $N \geq 3$, finding a first integral is not the end of the story, since its level sets will not be invariant curves but invariant "hypersurfaces", *i.e.* invariant subsets of dimension $N - 1$, on which there is room

[10] If you know what a submanifold is, you can check that a closed submanifold Σ of Ω is invariant if and only if, for each $p \in \Sigma$, the vector $f(p)$ belongs to the tangent space to Σ at p.

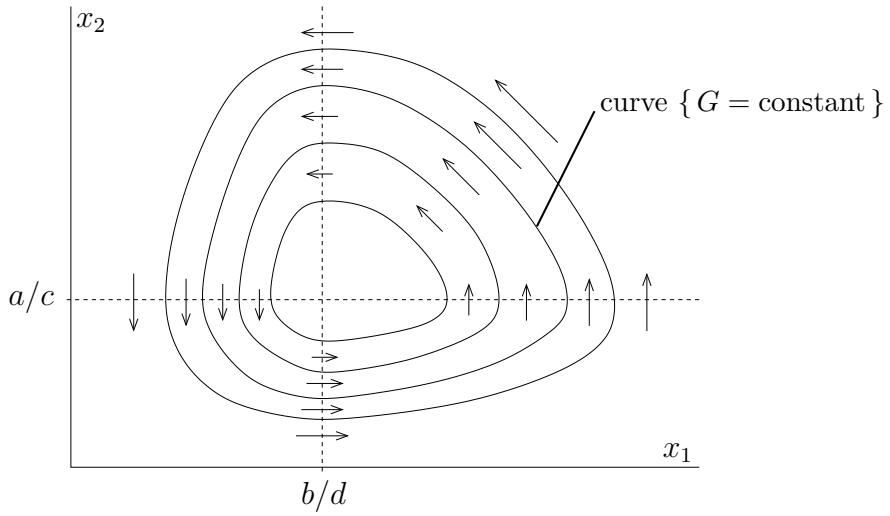

Figure 3: Periodic motions for Volterra's prey-predator system.

for all kinds of solutions. Still, it is worth finding as many first integrals as you can, or more generally invariant subsets of dimension as small as possible, in order to deal with "subsystems" of smaller dimension whenever possible—this may reduce drastically the complexity of the problem under study.

2.4 It is sometimes necessary to consider systems of **non-autonomous differential equations**, *i.e.* systems more general than (4) because the right-hand side also depends on time:

$$\left| \begin{array}{l} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2, \ldots, x_N, t) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2, \ldots, x_N, t) \\ \qquad \vdots \\ \dfrac{\mathrm{d}x_N}{\mathrm{d}t} = f_N(x_1, x_2, \ldots, x_N, t). \end{array} \right. \tag{8}$$

The corresponding vector-valued function

$$f = (f_1, \ldots, f_N) : \Omega \times \mathbb{R} \to \mathbb{R}^N,$$

is called a **non-autonomous vector field**, and the analogue of equation (5) is

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = f\big(\gamma(t), t\big). \tag{9}$$

Imagine for instance that we want to model the effect of fishing on the populations of sardines and sharks interacting according to (6). We may include in the right-hand side terms which contribute to decrease these populations, with

efficiency rates λ and μ:

$$\left|\begin{array}{rl}\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = & ax_1 - cx_1x_2 - \lambda x_1\\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = & -bx_2 + dx_1x_2 - \mu x_2.\end{array}\right.$$

If λ and μ are positive constants, this simply amounts to using new parameters $a-\lambda$ and $b+\mu$ instead of a and b (which modifies the location of the equilibrium point, for instance), but maybe we want to incorporate in the model periodic fluctuations of the fishing activity according to seasons, or non-periodic changes related with demography and economics... we then end up with

$$\left|\begin{array}{rl}\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = & \big(a-\lambda(t)\big)x_1 - cx_1x_2\\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = & -\big(b+\mu(t)\big)x_2 + dx_1x_2,\end{array}\right.$$

where the dependence on time in the right-hand side corresponds to external changes in the system (as opposed to internal changes of state modelled by the differential equations themselves).

2.5 The fundamental result of the theory is the Cauchy-Lipschitz theorem, which asserts the existence and unicity of the solution as soon as an **initial condition** is prescribed:

> *If Ω is an open subset of $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}^N$ is continuously differentiable, given any $p_0 \in \Omega$ and $t_0 \in \mathbb{R}$, there exists a unique solution $t \mapsto \gamma(t)$ of (9) such that $\gamma(t_0) = p_0$.*

The solution is not always defined for all $t \in \mathbb{R}$ (even in the autonomous case): depending on p_0 and t_0, there is a maximal interval I on which the solution is defined and satisfies the system of differential equations (of course this interval contains t_0).

Indeed, it may very well happen that the mobile point corresponding to the solution gains more and more speed and that it reaches infinity (or the boundary of Ω) in finite time. For example, consider the autonomous differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2$ with $\mathbb{R}$ as phase space: it is easily seen[11] that the only solution defined for all times is the constant solution $\gamma(t) = 0$ associated with the equilibrium point at the origin, while if $x_0 > 0$ for instance, the maximal solution γ such that $\gamma(0) = x_0$ is $t \in I \mapsto \gamma(t) = \dfrac{x_0}{1-tx_0}$ with $I = \left]-\infty, \dfrac{1}{x_0}\right[$.

This phenomenon never happens with linear systems, *i.e.* when the vector field f is defined by a matrix-valued function $t \mapsto A(t) = [a_{i,j}(t)]_{1\le i,j\le N}$ according

[11] In the autonomous case with $N = 1$, say $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$, the formula $t = t_0 + \int_{x_0}^{\gamma(t)} \frac{\mathrm{d}x'}{f(x')}$ is available, which only requires a quadrature, and we are left with the comparatively easy problem of inverting the relation $\gamma(t) \mapsto t$.

to the formulas

$$f_i(x_1, \dots, x_N, t) = a_{i,1}(t)x_1 + a_{i,2}(t)x_2 + \cdots + a_{i,N}(t)x_N, \qquad i = 1, \dots, N. \quad (10)$$

One can prove that all the solutions of such a linear system are defined for all times.

Sometimes, when dealing with a non-linear system, one can find a "trapping region": a closed subset $\mathcal{C}$ of the phase space with the property that at the boundary points of $\mathcal{C}$ the vector field points inside $\mathcal{C}$ (or is tangent to $\mathcal{C}$); in that situation, for all $p_0 \in \mathcal{C}$ and $t_0 \in \mathbb{R}$, the forward solution is defined without restriction on the whole of $[t_0, +\infty[$ and stays confined in $\mathcal{C}$ (because it cannot escape, so to speak).

3 The dynamical system point of view on differential equations

The theory of dynamical systems with continuous time can be described as the qualitative theory of differential equations; it aims at understanding all the possible behaviours of the system under study, although it is usually impossible to write explicit formulas for the solutions of differential equations. Rather than studying the solutions one by one, we try to define relevant geometric properties of the vector field; in fact, we already began, with the definition of first integrals, invariant subsets, trapping regions. Our main goal here will be to show the relation with discrete dynamical systems.

3.1 Let us consider a non-autonomous vector field $f : \Omega \times \mathbb{R} \to \mathbb{R}$ as in § 2.4. To simplify the discussion, we shall assume that all the solutions are defined for all times—such a vector field is said to be **complete**—, but this restriction is not essential.

The Cauchy-Lipschitz theorem allows us to define a family of maps from Ω in itself as follows. Given $t_0, t_1 \in \mathbb{R}$, we define the **flow map** $\Phi_{t_0,t_1} : \Omega \to \Omega$ by considering, for any point p, the solution γ with initial condition p at time t_0, *i.e.* $p_0 = \gamma(t_0)$, and following it until time t_1:

$$\Phi_{t_0,t_1}(p) = \gamma(t_1), \qquad \gamma \text{ unique solution such that } \gamma(t_0) = p.$$

It is obvious that $\Phi_{t_1,t_2} \circ \Phi_{t_0,t_1} = \Phi_{t_0,t_2}$ (flowing along a solution from time t_0 to time t_1, and then from time t_1 to time t_2, amounts to following the solution between times t_0 and t_2!), and that $\Phi_{t_0,t_0} = \mathrm{Id}$. As a consequence, each map Φ_{t_0,t_1} is invertible and $(\Phi_{t_0,t_1})^{-1} = \Phi_{t_1,t_0}$. (Moreover, since we always suppose our vector fields continuously differentiable, one can prove that these maps are continuously differentiable; this is a result on the regularity of the solutions with respect to initial conditions.)

Assume now that the vector field is autonomous, *i.e.* that $f : \Omega \to \mathbb{R}^N$ does not depend on time. Formula (20) in the appendix then shows that, for any constant $a \in \mathbb{R}$, the time-shift $t \mapsto a + t$ applied to a solution γ produces a parametrised

curve $t \mapsto \Gamma(t) = \gamma(a+t)$ which is also a solution: $\Gamma'(t) = \gamma'(a+t) = f(\gamma(a+t)) = f(\Gamma(t))$. If γ was the solution starting at p_0 at time $t_0 + a$, then Γ is the solution starting at the same point p_0 a moment before (if $a > 0$), at time t_0. It follows that $\Phi_{t_0,t_1} = \Phi_{a+t_0,a+t_1}$ (travelling between times t_0 and t_1 or doing it a moment later, between times $a + t_0$ and $a + t_1$, produces the same result; here, we use crucially the autonomous character of the vector field: the law of evolution of the system does not depend on the time at which you look at it).

Equivalently, we can say that the family of flow maps of an autonomous vector field can be written as a one-parameter family of maps: if we set[12]

$$\Phi^t = \Phi_{0,t} : \Omega \to \Omega, \qquad t \in \mathbb{R}$$

(*i.e.* we consider all initial conditions at time 0), we recover all the flow maps by $\Phi_{t_0,t_1} = \Phi^{t_1-t_0}$. We call Φ^t the **time-t map**; this yields a family of invertible maps, with $(\Phi^t)^{-1} = \Phi^{-t}$, which satisfies the so-called **one-parameter group property**

$$\Phi^0 = \mathrm{Id}, \qquad \Phi^s \circ \Phi^t = \Phi^{s+t}, \qquad s, t \in \mathbb{R}. \tag{11}$$

Thus, the solution with initial condition p_0 at time t_0 is $t \mapsto \gamma(t) = \Phi_{t_0,t}(p_0)$ in the non-autonomous case, and it can be written $t \mapsto \gamma(t) = \Phi^{t-t_0}(p_0)$ in the autonomous case.

Note: Given any family (φ_t) of maps $\Omega \to \Omega$ satisfying the one-parameter group property, we can define an autonomous vector field f by the formula

$$f(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi_t(p)\big)_{|t=0} \in \mathbb{R}^N, \qquad p \in \Omega. \tag{12}$$

We then observe that, for any t_0, the velocity vector $\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi_t(p)\big)_{|t=t_0}$ can be written $\frac{\mathrm{d}}{\mathrm{d}s}\big(\varphi_{s+t_0}(p)\big)_{|s=0}$ by formula (20) in the appendix, or $\frac{\mathrm{d}}{\mathrm{d}s}\big(\varphi_s(\varphi_{t_0}(p))\big)_{|s=0}$ by (11), and thus equals $f(\varphi_{t_0}(p))$ (by definition of f). This exactly means that φ_t is the time-t map of f. The vector field f is sometimes called "the infinitesimal generator" of φ_t.

3.2 A solution is, by definition, a parametrised curve $t \mapsto \gamma(t)$. Its range $\{\gamma(t), t \in \mathbb{R}\}$, which is a subset of the phase space, is often called an **orbit** (or a **trajectory**) of the vector field (*i.e.* we forget time-parametrisation when thinking about the orbit instead of the solution). Observe that, in the autonomous case, the trajectories of the system do not cross in the phase space Ω: if two solutions pass through the same point, they necessarily define the same curve, maybe with a time-shift. Indeed, if γ_1 and γ_2 are solutions with $\gamma_1(t_1) = \gamma_2(t_2) = p_*$, then $\gamma_1(t) = \Phi^{t-t_1}(p_*)$ and $\gamma_2(t) = \Phi^{t-t_2}(p_*)$ for all times.[13] We thus have a "foliation" of the phase space by solution curves.

[12]The notation Φ^t is reminiscent of the notation for the iterates of a discrete dynamical system, but here t is a real number, not necessarily an integer number!

[13]Consequently, in order that a solution γ be T-periodic, *i.e.* $\gamma(t+T) = \gamma(t)$ for all t, it is sufficient that $\gamma(t+T) = \gamma(t)$ for a particular value of t, for instance for $t = 0$.

This is not true in the non-autonomous case. But, according to the unicity statement in the Cauchy-Lipshitz theorem, we recover an analogous property when taking into account the time: if two solutions pass through the same point at the same time, they necessarily coincide for all times.

There is an abstract way of reconciling both cases: one can associate with any non-autonomous vector field (8)–(9) on Ω an autonomous vector field on the **extended phase space** $\tilde{\Omega} = \Omega \times \mathbb{R}$. The idea is to replace t in the right-hand side of (8) by an extra variable x_{N+1} subject to evolution law $\frac{\mathrm{d}x_{N+1}}{\mathrm{d}t} = 1$ (which will force it to coincide with time, up to a constant shift). We thus define $\tilde{f} : \tilde{\Omega} \to \mathbb{R}^{N+1}$ by

$$\left| \begin{array}{rcl} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} & = & f_1(x_1, x_2, \ldots, x_N, x_{N+1}) \\ & \vdots & \\ \dfrac{\mathrm{d}x_N}{\mathrm{d}t} & = & f_N(x_1, x_2, \ldots, x_N, x_{N+1}) \\ \dfrac{\mathrm{d}x_{N+1}}{\mathrm{d}t} & = & 1, \end{array} \right.$$

and we let the reader check that the corresponding autonomous flow satisfies $\tilde{\Phi}^{t_2-t_1}(p, t_1) = (\Phi_{t_1,t_2}(p), t_2)$ for all $(p, t_1) \in \tilde{\Omega}$. Thus,

> *any non-autonomous vector field on $\Omega \subset \mathbb{R}^N$ is equivalent to an autonomous vector field on $\tilde{\Omega} = \Omega \times \mathbb{R}$ with one more dimension.*

3.3 Similarly, any differential equation of order higher than 1 can be brought into the form of an autonomous system of differential equations of order 1 but of large enough dimension. For instance, laws of physics often correspond to second-order differential equations for one or several variables ("degrees of freedom"), and experience indeed tells us that the position of a mechanical system at a given instant is not sufficient to determine its motion, we need to specify the velocity at the same instant. Here are two classical examples.

The equation for the **pendulum**, which is discussed at the beginning of [Ch] (see the numerous figures therein), can be written

$$m\ell \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -mg \sin x,$$

where the position of the pendulum is determined by the angle x counted counterclockwise from the lower position, m is the mass of the pendulum, ℓ the length of the string, and g the gravitational constant at Earth's surface. Introducing the angular velocity $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, we get the system

$$\left| \begin{array}{rcl} \dfrac{\mathrm{d}x}{\mathrm{d}t} & = & v \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} & = & -\dfrac{g}{\ell} \sin x. \end{array} \right. \tag{13}$$

The phase space is thus $\Omega = \mathbb{R} \times \mathbb{R}$, or rather $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the circle with angular coordinate x defined modulo 2π. We shall return to this simple model later.

Another famous example is the so-called **three-body problem**. Consider three masses m_1, m_2, m_3 with positions $x^{(1)}, x^{(2)}, x^{(3)}$ in $\mathbb{R}^3$ (which amounts to 9 real variables $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_1^{(3)}, x_2^{(3)}, x_3^{(3)}$), subject to Newton's law of gravitation only: we get 9 second-order differential equations, which can be written as three equations in $\mathbb{R}^3$:

$$\left|\begin{array}{l} m_1 \dfrac{\mathrm{d}^2 x^{(1)}}{\mathrm{d}t^2} = G m_1 m_2 \dfrac{x^{(2)} - x^{(1)}}{\|x^{(2)} - x^{(1)}\|^3} + G m_1 m_3 \dfrac{x^{(3)} - x^{(1)}}{\|x^{(3)} - x^{(1)}\|^3} \\ m_2 \dfrac{\mathrm{d}^2 x^{(2)}}{\mathrm{d}t^2} = G m_2 m_3 \dfrac{x^{(3)} - x^{(2)}}{\|x^{(3)} - x^{(2)}\|^3} + G m_2 m_1 \dfrac{x^{(1)} - x^{(2)}}{\|x^{(1)} - x^{(2)}\|^3} \\ m_3 \dfrac{\mathrm{d}^2 x^{(3)}}{\mathrm{d}t^2} = G m_3 m_1 \dfrac{x^{(1)} - x^{(3)}}{\|x^{(1)} - x^{(3)}\|^3} + G m_3 m_2 \dfrac{x^{(2)} - x^{(3)}}{\|x^{(2)} - x^{(3)}\|^3}. \end{array}\right.$$

To obtain a system of first-order differential equations, we need to introduce 9 more variables, namely the components of the velocities $v^{(i)} = \dfrac{\mathrm{d}x^{(i)}}{\mathrm{d}t}$. We get an autonomous vector field in dimension 18, which splits into six $\mathbb{R}^3$-components:

$$\left|\begin{array}{l} \dfrac{\mathrm{d}x^{(1)}}{\mathrm{d}t} = v^{(1)} \\ \dfrac{\mathrm{d}v^{(1)}}{\mathrm{d}t} = G m_2 \dfrac{x^{(2)} - x^{(1)}}{\|x^{(2)} - x^{(1)}\|^3} + G m_3 \dfrac{x^{(3)} - x^{(1)}}{\|x^{(3)} - x^{(1)}\|^3} \\ \dfrac{\mathrm{d}x^{(2)}}{\mathrm{d}t} = v^{(2)} \\ \dfrac{\mathrm{d}v^{(2)}}{\mathrm{d}t} = \dots \\ \qquad \vdots \end{array}\right.$$

Here the phase space is $\big(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta\big) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, where

$$\Delta = \{\, (x^{(1)}, x^{(2)}, x^{(3)}) \mid x^{(1)} = x^{(2)} \text{ or } x^{(2)} = x^{(3)} \text{ or } x^{(3)} = x^{(2)} \,\}$$

is the "collision set". This system admits some first integrals which help to reduce the dimension (among which the three components of the total "impulsion" $m_1 v^{(1)} + m_2 v^{(2)} + m_3 v^{(3)}$ and the total "energy" defined in § 3.5), but its behaviour is far from being well understood (with only two bodies, on the contrary, all the solutions can be described satisfactorily—this is the so-called Kepler problem).

Keep in mind that *one can always work with an autonomous vector field*, provided the phase space is suitably defined, and that *the time-t maps Φ^t are then sufficient to describe all the solutions, each point in the phase space determining uniquely a solution which goes through it at time* 0.

3.4 As in the case of discrete dynamical systems, some systems are simpler than the others regardless of the dimension: we say that we have a **linear system of differential equations with constant coefficients** when the vector field f is autonomous and defined through a constant matrix $A = [a_{i,j}]_{1\leq i,j\leq N}$ by the formula

$$f : x \in \mathbb{R}^N \mapsto A \cdot x \in \mathbb{R}^N. \tag{14}$$

The corresponding system of differential equations has the same form as in (10), but this time the coefficients $a_{i,j}$ are constant, *i.e.* we have an autonomous linear system. The salient feature of linear systems, autonomous or not, is that the flow maps Φ_{t_0,t_1} are linear maps from $\mathbb{R}^N$ into $\mathbb{R}^N$ (as is easily checked by using the unicity statement in the Cauchy-Lipschitz theorem and the characterisation of linear maps as those which send a linear combination of vectors to the linear combination of their images with the same coefficients). In the autonomous linear case, we have moreover the formula[14]

$$\Phi^t = \exp(tA),$$

where $\exp(\,.\,)$ denotes matrix exponentiation:

$$\exp(B) \cdot x = x + B \cdot x + \frac{1}{2!}B^2 \cdot x + \frac{1}{3!}B^3 \cdot x + \cdots$$

for any matrix B and vector x.

For an autonomous linear system, the phenomenology is similar to what was alluded to in §§ 1.5–1.7, the most important thing being the eigenvalues $\lambda_1, \ldots, \lambda_N$ of the matrix A, which yield eigenvalues $\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_N}$ for the linear map Φ^t. If an eigenvalue λ_i is negative, it is associated with contraction along the corresponding directions: there exists nonzero vectors x such that $A \cdot x = \lambda_i\, x$, which yields $\Phi^t(x) = \mathrm{e}^{t\lambda_i} x \xrightarrow[t\to+\infty]{} 0$. Similarly, a positive eigenvalue λ_i is associated with dilatation. More generally, for complex eigenvalues, what matters if the sign of the real part: the contraction or dilatation may be combined with rotation if the imaginary part is nonzero.

Purely imaginary eigenvalues are associated with oscillatory behaviour. We can illustrate this with the small oscillations of the pendulum around its lower equilibrium point. Consider the case where the angle x_0 is close to 0 and we are given an initial velocity v_0 which is not too large. Since $\sin x \sim x$ for small $|x|$, we can model this by the linear system

$$\left| \begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= v \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{g}{\ell}x \end{aligned} \right. \tag{15}$$

instead of the nonlinear system (13). The corresponding matrix is $A = \left(\begin{smallmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{smallmatrix}\right)$. Standard linear algebra shows that $\exp(tA) = \left(\begin{smallmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{smallmatrix}\right)$ with $\omega = \sqrt{g/\ell}$

[14] Beware that no such formula exists for non-autonomous linear systems, which can be much harder to study than the autonomous ones.

(the eigenvalues of A are $\mathrm{i}\,\omega$ and $-\mathrm{i}\,\omega$, those of $\exp(tA)$ are $\mathrm{e}^{\mathrm{i}\,\omega t}$ and $\mathrm{e}^{-\mathrm{i}\,\omega t}$), and that the solutions travel along ellipses. All the solutions are periodic with the same period $T = \frac{2\pi}{\omega}$, the projection onto the x-axis oscillating periodically between the two positions corresponding to the vanishing of v.

The passage from system (13) to (15) is called linearisation, because it consists in replacing the components of the right-hand side by a linear approximation. However, we cannot expect that this approximation make sense in the whole phase space. For instance, the origin of $\mathbb{R}^2$ is the only equilibrium point of the linear system, whereas the nonlinear system has another one, namely the point $(\pi, 0)$ which corresponds to the upper equilibrium position of the pendulum.

Before proceeding with a more complete description of the solutions of the pendulum (13), let us end this paragraph with a simple example showing how "non-homogeneous" linear equations may appear in the context of autonomous vector fields. If we consider the system in $\mathbb{R}^3$

$$\left| \begin{array}{l} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -y + x^2 \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = z + x^2 \end{array} \right. \tag{16}$$

and try to compute the solution $\gamma(t) = (x(t), y(t), z(t))$ with a given initial condition (x_0, y_0, z_0) at time 0, the first equation immediately yields $x(t) = x_0\,\mathrm{e}^{-t}$. We are thus left with two independent problems

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y + x_0^2\,\mathrm{e}^{-2t}, \qquad y(0) = y_0$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z + x_0^2\,\mathrm{e}^{-2t}, \qquad z(0) = z_0.$$

One can then easily check that $y(t) = y_0\,\mathrm{e}^{-t} + x_0^2(\mathrm{e}^{-t} - \mathrm{e}^{-2t})$ and $z(t) = z_0\,\mathrm{e}^{t} + \frac{x_0^2}{3}(\mathrm{e}^{t} - \mathrm{e}^{-2t})$ (use for instance the auxiliary unknown function $u(t) = z(t)\,\mathrm{e}^{-t}$ to compute $z(t)$), this is one of the rare cases of an explicit nonlinear time-t map:

$$\Phi^t(x, y, z) = \left(x\,\mathrm{e}^{-t},\ y\,\mathrm{e}^{-t} + x^2(\mathrm{e}^{-t} - \mathrm{e}^{-2t}),\ z\,\mathrm{e}^{t} + \frac{x^2}{3}(\mathrm{e}^{t} - \mathrm{e}^{-2t}) \right).$$

3.5 Let us return to the pendulum described by system (13) and try to understand its **phase portrait**, *i.e.* the foliation of the phase space by the orbits. Similarly to what happened in the example discussed in § 2.3, there is a first integral, the level sets of which are the orbits, except that this time there will be a singular level set.

This first integral of the pendulum is the function

$$H : (x, v) \in \mathbb{S}^1 \times \mathbb{R} \ \mapsto \ H(x, v) = \frac{1}{2}v^2 + \frac{g}{\ell}(1 - \cos x) = \frac{1}{2}v^2 + \frac{2g}{\ell}\sin^2(x/2).$$

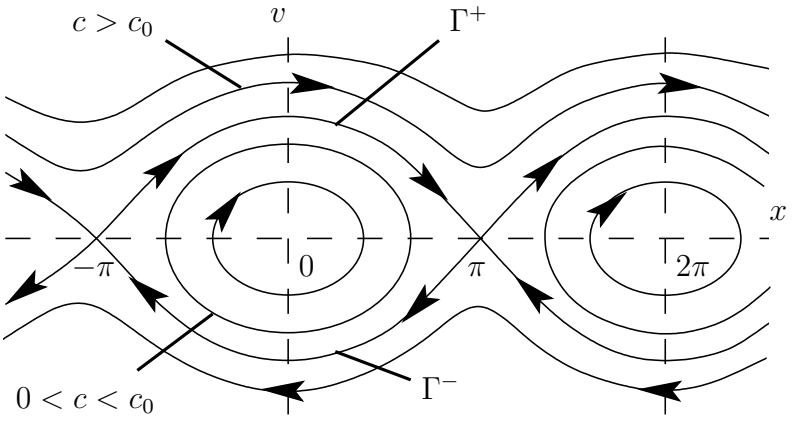

Figure 4: Phase portrait of the pendulum.

Indeed, you can check that $v\dfrac{\partial H}{\partial x}(x,v) - \dfrac{g}{\ell}(\sin x)\dfrac{\partial H}{\partial v}(x,v) = 0$. This function is, up to a multiplicative constant, the **total energy**, sum of a kinetic part and a potential. The level sets of the energy function H are represented on figure 4.

The level $\{\, H = 0 \,\}$ consists of only one point, the equilibrium point $(0,0)$ which corresponds to the lower position of the pendulum with zero velocity. For $0 < c < \frac{2g}{\ell}$, the level $\{\, H = c \,\}$ is a closed curve surrounding the origin, which is well approached by the ellipsis $\{\, \frac{1}{2}v^2 + \frac{g}{2\ell}x^2 = c \,\}$ for small c: this is the linear approximation for the small oscillations; but when the oscillations have larger amplitude, the period of the motion gets larger (contrarily to the linearised system, in which all the solutions had the same period). For $c > \frac{2g}{\ell}$, the velocity v never vanishes along the level curve $\{\, H = c \,\}$, this corresponds to the periodic motions in which the pendulum periodically makes a complete clockwise or anti-clockwise turn (the angular coordinate x moves always in the same direction without stopping, but we view this as a periodic evolution in $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$). When c increases, the period of these revolutions gets smaller because the speed $|v|$ is larger on average.

The value $c_0 = \frac{2g}{\ell}$ corresponds to a limit case, which separates oscillation motions from revolution motions. Using $\sin^2 x = 1 - \cos^2 x$, we can write the corresponding level set as

$$\{\, H = c_0 \,\} = \{\, (x,v) \mid \tfrac{1}{2}v^2 - \tfrac{2g}{\ell}\cos^2(x/2) = 0 \,\},$$

which is the union of two curves $\{\, (x,v) \in \,]-\pi,\pi[\times \mathbb{R} \mid v = 2\sqrt{\tfrac{g}{\ell}}\cos(x/2) \,\}$ and $\{\, (x,v) \in \,]-\pi,\pi[\times \mathbb{R} \mid v = -2\sqrt{\tfrac{g}{\ell}}\cos(x/2) \,\}$ called "separatrices".

This singular level set consists of 3 orbits: the equilibrium point $(\pi,0)$ (identified with $(-\pi,0)$), an orbit $\Gamma^+ = \{\, (x,v) \in \,]-\pi,\pi[\times \mathbb{R} \mid v = 2\sqrt{\tfrac{g}{\ell}}\cos(x/2) \,\}$, which is for instance the orbit followed when the pendulum starts at the lower position with the precise velocity $v_0 = 2\sqrt{\frac{g}{\ell}}$ which will make it reach the upper position in infinite time, and an orbit Γ^- symmetric of Γ^+. Apart from the equilibrium points, these are the only non-periodic solutions;[15] they are positively and

[15]You can also consider them as periodic with period $T = +\infty$, for the period of the solution

negatively asymptotic to the "unstable" equilibrium $(\pi, 0)$.

The three-body problem too admits as first integral the total energy $H = K+V$, sum of the **kinetic energy**

$$K = \frac{1}{2}m_1\|v^{(1)}\|^2 + \frac{1}{2}m_2\|v^{(2)}\|^2 + \frac{1}{2}m_3\|v^{(3)}\|^2$$

and of the **potential**

$$V = -\frac{Gm_1m_2}{\|x^{(1)} - x^{(2)}\|} - \frac{Gm_2m_3}{\|x^{(2)} - x^{(3)}\|} - \frac{Gm_3m_1}{\|x^{(3)} - x^{(1)}\|}.$$

Other first integrals are known, but not enough to understand fully the orbits of this system!

3.6 Not all the systems admit a first integral. For instance, one can transform the pendulum equation into

$$m\ell\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -mg\sin x - \rho\frac{\mathrm{d}x}{\mathrm{d}t}$$

with $\rho > 0$, in order to model friction as a damping force proportional to velocity. We let the reader write the corresponding autonomous vector field and study its phase portrait (see for instance [Ch], § 2, or [HW], Part 2, p. 32). Not suprisingly, the energy function H is no longer a first integral, but it decays along the solutions (the system "loses energy"). Apart from the two equilibrium points $p_{\text{low}} = (0, 0)$ and $p_{\text{up}} = (\pi, 0)$, and an exceptional solution which tends to p_{up}, all the solutions $\gamma(t)$ tend to p_{low}. As a consequence, any first integral G must be constant, for $G(\gamma(0)) = G(\gamma(t)) \xrightarrow[t\to+\infty]{} G(p_{\text{low}})$.

So far, we have seen examples of solutions which are periodic or attracted by an equilibrium point. It may happen that a periodic solution attracts all the nearby solution; it is then called a **limit cycle**. The famous Van der Pol equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \mu(x^2 - 1)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0,$$

which models an electrical circuit, possesses such a limit cycle.

Here is a simpler example:

$$\left|\begin{array}{l} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y + \alpha(1 - x^2 - y^2)x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + \alpha(1 - x^2 - y^2)y, \end{array}\right.$$

with $\alpha > 0$. There is a unique equilibrium point, at the origin. For any solution $\gamma(t) = (x(t), y(t))$ with an initial condition different from the origin, setting $r(t) = x(t)^2 + y(t)^2$, we get $r'(t) = 2\alpha(1 - r(t))r(t)$. This shows that the unit circle

$t \mapsto \Phi^t(p)$ tends to $+\infty$ as $\mathrm{dist}(p, \Gamma^+ \cup \Gamma^-) \to 0$.

$\{ x^2 + y^2 = 1 \}$ is invariant, and it is obviously a periodic orbit (along which the angular velocity is constant), but this also shows that all the solutions $\gamma(t)$ are attracted by the unit circle, since $r'(t) > 0$ when $r(t) < 1$ and $r'(t) < 0$ when $r(t) > 1$. This should be sufficient to help you to draw the phase portrait.

3.7 It is time to discuss more deeply the relation between vector fields and discrete dynamical systems. We shall indicate three situations in which a vector field f, autonomous or not, gives rise to a map P, in such a way that iterating the map corresponds to flowing along the solutions of f for a certain time. This will give tools to transfer results on maps to vector fields.

a) We have already associated with an autonomous vector field $f : \Omega \to \mathbb{R}^N$ the family of time-t maps (Φ^t). This is the most obvious case: for a given $\tau > 0$, we can set $P = \Phi^\tau$.

Thus, for all $p \in \Omega$ and $n \in \mathbb{Z}$, $P^n(p) = \Phi^{n\tau}(p)$. In particular, P is invertible, with $P^{-1} = \Phi^{-\tau}$. Fixed points of Φ^τ are initial conditions of τ-periodic solutions of f, according to footnote 13. More generally, a k-periodic orbit of P gives rise to a $k\tau$-periodic orbit of f.

In this case, the vector field and the map are defined[16] in the same set Ω.

b) Let us now consider a non-autonomous vector field $f : \Omega \times \mathbb{R} \to \mathbb{R}^N$, but with a periodic dependence on time:

$$f(p, t + T) = f(p, t), \qquad p \in \Omega,\ t \in \mathbb{R},$$

where $T > 0$ is fixed.

Beware that this does not imply that the solutions of f are T-periodic! But there is a subtler kind of periodicity in the flow maps:

$$\Phi_{t_0,t_1} = \Phi_{t_0+T,t_1+T}, \qquad t_0, t_1 \in \mathbb{R}.$$

This is easy to check: Let $p \in \Omega$, $t_0 \in \mathbb{R}$, $\gamma(t) = \Phi_{t_0,t}(p)$, $\Gamma(t) = \Phi_{t_0+T,t}(p)$. We want to prove that $\gamma(t) = \Gamma(t+T)$ and we already have $\gamma(t_0) = p = \Gamma(t_0+T)$. On the other hand, according to formula (20) in the appendix, the velocity vector of the parametrised curve $t \mapsto \Gamma(t+T)$ is $\Gamma'(t+T) = f(\Gamma(t+T), t+T) = f(\Gamma(t+T), t)$ by assumption, thus this parametrised curve is the solution with initial condition p at time t_0.

As a particular case, we have

$$\cdots = \Phi_{t_0-T,t_0} = \Phi_{t_0,t_0+T} = \Phi_{t_0+T,t_0+2T} = \Phi_{t_0+2T,t_0+3T} = \cdots$$

Let us define $P = \Phi_{t_0,t_0+T}$ for a given t_0. This is called the **Poincaré map** associated with the section $\{ t \equiv t_0 \}$. This map is invertible, with $P^{-1} = \Phi_{t_0+2T,t_0+T} = \Phi_{t_0+T,t_0} = \Phi_{t_0,t_0-T} \ldots$ For any point p and integer n, we have

$$P^n(p) = \Phi_{t_0+(n-1)T,t_0+nT} \circ \Phi_{t_0+(n-2)T,t_0+(n-1)T} \circ \cdots \circ \Phi_{t_0,t_0+T}(p) = \Phi_{t_0,t_0+nT}(p).$$

[16] Observe that the construction only requires that all the initial conditions in Ω give rise to a forward solution of f defined for all positive times: this is sufficient to have $P^n(p) = \Phi^{n\tau}(p)$ for all $n \geq 0$, even if the vector field is not complete.

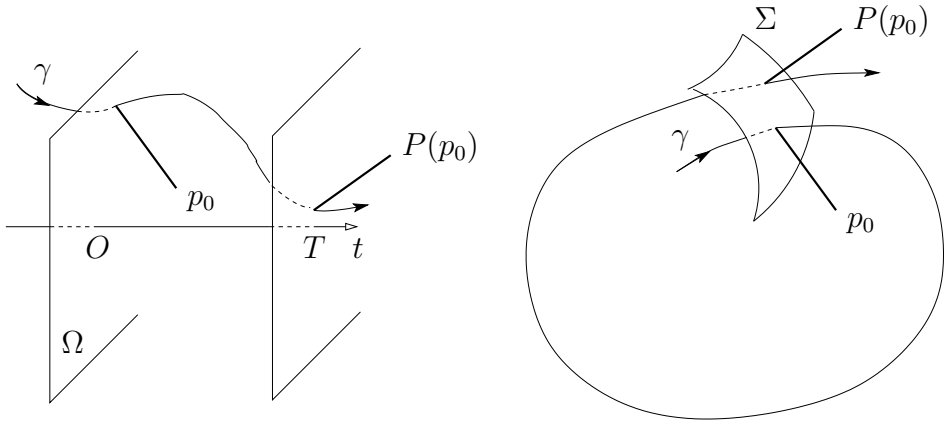

Figure 5: The Poincaré map P for a periodic vector field (left) and for a transversal section of an autonomous vector field (right).

Hence, iterating P makes sense and corresponds to looking at the solution $\gamma(t) = \Phi_{t_0,t}(p)$ at times $t = t_0 + nT$, as with a stroboscope.

If p_0 is a k-periodic point of P, the solution of f with initial condition p_0 at time t_0 is kT-periodic.

Here, the extended phase space $\Omega \times \mathbb{R}$ of the vector field (in which the solutions are pairwise disjoint curves) has one more dimension than the phase space Ω of the Poincaré map.

Note: If moreover $f(p,t)$ is linear in $p = (x_1, \ldots, x_N) \in \mathbb{R}^N$ (*cf.* (10) in § 2.5), each map Φ_{t_0,t_1} is linear, in particular the Poincaré map is a linear map and can be analysed in the spirit of §§ 1.5–1.7 (first look at the eigenvalues and try to diagonalise...), leading to the so-called Floquet theory.

c) We finally consider an autonomous vector field $f : \Omega \to \mathbb{R}^N$ and a **transversal section** Σ. This means that Σ is a $(N-1)$-dimensional surface in the N-dimensional set Ω, with the property that at each point p of Σ the vector field $f(p)$ is not tangent to Σ.

It is sometimes possible to define the **Poincaré return map** associated with the section Σ: the idea is that when an initial condition p_0 is chosen on Σ, the corresponding solution curve $\gamma(t) = \Phi^t(p_0)$ will necessarily leave Σ (by the transversality assumption), but it may return to Σ after an excursion in $\Omega \setminus \Sigma$; the correspondence between p_0 and the first return point $p_1 = \gamma(\tau) \in \Sigma$ (for a certain $\tau > 0$ *which depends on* p_0) then defines a map from Σ to Σ.

More formally, for a point $p_0 \in \Sigma$, we define (whenever possible) $P(p_0) = \Phi^{\tau(p_0)}(p_0)$ where the return time $\tau(p_0) > 0$ is characterised by $\Phi^{\tau(p_0)}(p_0) \in \Sigma$ and $\Phi^{\tau}(p_0) \notin \Sigma$ for $0 < \tau < \tau(p_0)$.

Iterating the map P still corresponds to flowing along a solution of f, extracting from the solution a discrete sequence of points, but the peculiarity of this construction is that the times at which the solution passes through these points are not necessarily equally spaced. A fixed point (or a periodic point) p_* of P

corresponds to a solution γ_* which is periodic, but the period is not determined *a priori*, it is imposed by the geometry itself.

One can prove that conversely, if $t \mapsto \gamma_*(t)$ is a periodic solution of f, the above construction is possible in any small enough section Σ which is transversal to γ_* at a point of γ_*.

The interest of the return map is that its phase space has one less dimension than the phase space of the autonomous vector field f.

The above case b), involving a non-autonomous vector field f depending periodically on time, was a particular case of this construction: if we take $\Omega \times (\mathbb{R}/T\mathbb{Z})$ as extended phase, *i.e.* we consider the last variable x_{N+1} as defined modulo T, we get an autonomous vector field $\tilde{f}$ for which $\{\, x_{N+1} \equiv t_0 \,\}$ is a transversal section with a well-defined return map (because $\frac{\mathrm{d}x_{N+1}}{\mathrm{d}t} = 1$).

4 The stable manifold theorem

4.1 A standard approach to many systems which look difficult to analyse is to view them as perturbations of simpler systems: sometimes, by neglecting a few terms in the law of evolution, we get a system for which the phase portrait is known, or at least some solutions are well understood, and one tries then to justify the approximation, *i.e.* to prove the existence of features in the original complicated system which are similar, maybe with a slight deformation, to the known features of the simpler system.

The first instance of this approach is **linearisation**: when you have an equilibrium point for a non-linear system, you can try to approach the nearby solutions by the solutions of a linear system, as we already did in § 3.4 for the differential equation of the pendulum.

4.2 In the case of a discrete dynamical system F with a fixed point p_* (we assume that F is a differentiable map from Ω to Ω, where Ω is an open subset of $\mathbb{R}^d$), the **linearised system** is defined as the differential $\mathrm{D}F(p_*)$, viewed as a linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$ to be iterated. In view of formula (18) of the appendix, this indeed corresponds to a linear approximation when performing the change of coordinates $x = p_* + X$:

$$F(p_* + X_n) \approx p_* + X_{n+1}, \qquad X_{n+1} = \mathrm{D}F(p_*) \cdot X_n$$

(remember that $F(p_*) = p_*$).

Note: Observe that $\mathrm{D}(F^n)(p_*) = \big(\mathrm{D}F(p_*)\big)^n$, *i.e.* the linearisation of the n^{th} iterate is the n^{th} iterate of the linearisation. This is a consequence of the chain rule (§ 5.6 in the appendix); more generally, if $\{p_n = F^n(p_0)\}$ is any orbit, $\mathrm{D}(F^n)(p_0) = \mathrm{D}F(p_{n-1}) \circ \mathrm{D}F(p_{n-2}) \circ \cdots \circ \mathrm{D}F(p_0)$.

We say that $A = \mathrm{D}F(p_*)$ is a **hyperbolic matrix**, or that p_* is a **hyperbolic fixed point** of F, when all the (possibly complex) eigenvalues of this matrix have modulus $\neq 1$.

A standard result of linear algebra then states that each vector X of $\mathbb{R}^d$ can be written, in a unique fashion, as the sum of two vectors $X^{(s)}$ and $X^{(u)}$ such that

$$\|A \cdot X^{(s)}\| \leq \kappa \|X^{(s)}\|, \quad \|A^{-1} \cdot X^{(u)}\| \leq \kappa \|X^{(u)}\|,$$

where κ is a fixed constant which satisfies $0 < \kappa < 1$ and which depends only on the matrix $\mathrm{D}F(p_*)$ (possibly with a modification of the definition of the norm $\|\,.\,\|$, but this is innocuous).

We have already seen such a situation in § 1.6, where the matrix A for that example was proved to have eigenvalues 4 and $\frac{1}{2}$ with the help of a linear change of coordinates. Indeed, in the diagonalising coordinates, the decomposition of an arbitrary vector X was simply

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ X_2 \end{pmatrix} + \begin{pmatrix} X_1 \\ 0 \end{pmatrix}, \qquad D \cdot \begin{pmatrix} 0 \\ X_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ X_2 \end{pmatrix}, \quad D^{-1} \cdot \begin{pmatrix} X_1 \\ 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} X_1 \\ 0 \end{pmatrix}.$$

In the original coordinates (we were then using the letter x), this corresponded to the decomposition of $\mathbb{R}^2$ into the direct sum $E^{(s)} \oplus E^{(u)}$ of two invariant lines, the maps $x \mapsto x^{(s)}$ and $x \mapsto x^{(u)}$ being the associated projectors.

For a general hyperbolic matrix too, we have a decomposition $\mathbb{R}^d = E^{(s)} \oplus E^{(u)}$ into two invariant vector subspaces, called the "stable" and "unstable" subspaces (maybe of higher dimensions than a line; for instance, in the case of the matrix $\mathcal{A}$ of § 1.7, one finds that $E^{(s)}$ is the 2-dimensional vector subspace spanned by the first two vectors of the standard basis of $\mathbb{R}^4$, and that $E^{(u)}$ is the 2-dimensional vector subspace spanned by the last two vectors).

The stable subspace consists of the vectors X such that $X = X^{(s)}$. For such vectors, we have by induction

$$\|A^n \cdot X\| \leq \kappa^n \|X\| \xrightarrow[n \to +\infty]{} 0.$$

Similarly, for vectors in the unstable subspace, $X = X^{(u)}$ and

$$\|A^{-n} \cdot X\| \leq \kappa^n \|X\| \xrightarrow[n \to +\infty]{} 0.$$

4.3 Among the orbits of a hyperbolic matrix A, we thus have two remarkable families:
– If the initial condition X_0 belongs to $E^{(s)}$, all past and future iterates stay in $E^{(s)}$ (because this subspace is invariant), and the future iterates X_n converge to 0.
– If the initial condition X_0 belongs to $E^{(u)}$, all past and future iterates stay in $E^{(s)}$, and the past iterates X_{-n} converge to 0.

It turns out that a nonlinear version of this phenomenon takes place for any discrete dynamical system with a hyperbolic fixed point—this is the content of the **stable manifold theorem**:

If $F : \Omega \to \Omega$ has a hyperbolic fixed point p_*, the formulas

$$\mathcal{W}^{(s)} = \{\, p \in \Omega \mid F^n(p) \xrightarrow[n \to +\infty]{} p_* \,\}, \quad \mathcal{W}^{(u)} = \{\, p \in \Omega \mid F^{-n}(p) \xrightarrow[n \to +\infty]{} p_* \,\}$$

define two invariant subsets, which are submanifolds of Ω containing p_*, respectively tangent to $E^{(s)}$ and $E^{(u)}$ at p_*.

The fact that $\mathcal{W}^{(s)}$ is a submanifold tangent to $E^{(s)}$ at p_* simply means the following:

We can use $(X^{(s)}, X^{(u)}) \in E^{(s)} \times E^{(u)}$ as "coordinates" representing the point $p = p_* + X^{(s)} + X^{(u)}$, where $X^{(s)}$ is a vector which requires $d^{(s)}$ real numbers to be determined if $d^{(s)}$ is the dimension of $E^{(s)}$, and $X^{(u)}$ is a vector of dimension $d^{(u)} = \dim E^{(u)}$. In these coordinates, the fixed point p_* corresponds to $(0,0)$, the linear approximation of F reads $(X^{(s)}, X^{(u)}) \mapsto (A{\cdot}X^{(s)}, A{\cdot}X^{(u)})$ with $A = \mathrm{D}F(p_*)$, and $E^{(s)}$ corresponds to $\{\, X^{(u)} = 0 \,\}$.

Well, locally, in these coordinates, $\mathcal{W}^{(s)}$ corresponds to $\{\, X^{(u)} = \varphi(X^{(s)}) \,\}$, where φ is a differentiable map $E^{(s)} \to E^{(u)}$ with $\mathrm{D}\varphi(0) = 0$. Therefore, we can think of this set as of a $d^{(s)}$-dimensional surface, the linear approximation of which is $\{\, X^{(u)} = 0 \,\} = E^{(s)}$.

Similarly, $\mathcal{W}^{(u)}$ locally corresponds to $\{\, X^{(s)} = \psi(X^{(u)}) \,\}$, where ψ is a differentiable map $E^{(u)} \to E^{(s)}$ with $\mathrm{D}\psi(0) = 0$, therefore the linear approximation of this $d^{(u)}$-dimensional surface is $\{\, X^{(s)} = 0 \,\} = E^{(u)}$.

The sets $\mathcal{W}^{(s)}$ and $\mathcal{W}^{(u)}$ are called the **stable and unstable manifolds** of p_* for F; they are sometimes denoted $\mathcal{W}^{(s)}(p_*, F)$ and $\mathcal{W}^{(u)}(p_*, F)$. The reader is referred to any dynamical system textbook for the proof of the aforementioned theorem.

4.4 Here is an example in $\mathbb{R}^3$: the map

$$F : (x, y, z) \mapsto (x', y', z') = (\tfrac{1}{2}x, \tfrac{1}{2}y + \tfrac{1}{4}x^2, 2z + \tfrac{7}{12}x^2)$$

clearly has a hyperbolic fixed point at the origin, with $\mathrm{D}F(0){\cdot}X = (\frac{1}{2}X_1, \frac{1}{2}X_2, 2X_3)$, with eigenvalues $\frac{1}{2}$ and 2, and with a 2-dimensional subspace $E^{(s)} = \{\, z = 0 \,\}$ and a 1-dimensional subspace $E^{(u)} = \{\, x = y = 0 \,\}$.

One can see that $E^{(u)}$ is invariant not only by $\mathrm{D}F(0)$, but also by F itself, therefore $\mathcal{W}^{(u)} = E^{(u)}$; on this set, the dynamics of F^{-1} is simply $(0, 0, z) \mapsto (0, 0, \frac{1}{2}z)$, which yields convergence of the backward orbits to the origin, as should be.

For this particular map, the stable manifold can be computed exactly:

$$\mathcal{W}^{(s)} = \{\, (x, y, z) \in \mathbb{R}^3 \mid z + \frac{x^2}{3} = 0 \,\},$$

i.e. $\varphi(X_1, X_2) = -\frac{1}{3}X_1^2$ in the above notations. Indeed, $z = -x^2/3$ implies that $z' = -\frac{2}{3}x^2 + \frac{7}{12}x^2 = -\frac{1}{12}x^2$, thus $z' = -x'^2/3$ (since $x' = x/2$): this establishes the invariance of the set $\{\, z = -x^2/3 \,\}$. The forward orbits can be computed:

$$F^n(x, y, -\tfrac{1}{3}x^2) = \left(\tfrac{1}{2^n}x,\, \tfrac{1}{2^n}y + \left(\tfrac{1}{2^n} - \tfrac{1}{4^n}\right)x^2,\, -\tfrac{1}{3}\left(\tfrac{1}{2^n}x\right)^2\right),$$

and they converge to the origin, as should be.

However, this example is quite exceptional. Even in dimension $d = 2$, the situation can be much more intricate. For instance the standard map F, which was defined by (1) at the beginning, has a hyperbolic fixed point at $(0, 0) \in \mathbb{S}^1 \times \mathbb{R}$. The reader may wonder where the stable and unstable manifolds are on figure 1... Well, they are on the picture, but these curves are so complicated that they seem to create a chaotic region which contains the fixed point. This is related to the fact that we have not been very precise when using the word "locally" to explain the stable manifold theorem: in fact, it was only a piece of $\mathcal{W}^{(s)}$, called the local stable manifold $\mathcal{W}^{(s)}_{loc}$ which could be repesented as a nice $d^{(s)}$-dimensional surface $\{ X^{(u)} = \varphi(X^{(s)}) \}$, and this piece is forward-invariant: $p \in \mathcal{W}^{(s)}_{loc}$ implies $F(p) \in \mathcal{W}^{(s)}_{loc}$; from this, the whole stable manifold can be recovered by adding the points of $F^{-1}(\mathcal{W}^{(s)}_{loc})$, *i.e.* the points of which the first iterate is in $\mathcal{W}^{(s)}_{loc}$, and also the points of $F^{-2}(\mathcal{W}^{(s)}_{loc})$, etc. Proceeding this way, at each step the set of points we add is a nice $d^{(s)}$-dimensional manifold (because F^{-1} is a differentiable invertible map), but globally, the set is not so regular: in the case of the standard map, it turns out that after a few steps of this process the curve you must add is a very long and sinuous one.

We shall not discuss further this situation, which is sometimes called **homoclinic tangle** and which conceals fascinating features—see *e.g.* § 4 of [Ch].

4.5 Perhaps the reader wonders how the orbits were computed in the first example of the previous paragraph. In fact, this is a particular case of a formula that we have already obtained at the end of § 3.4! The map F of the example was nothing but the time-τ map of the autonomous vector field (16) with $\tau = \log 2$, so we know how to iterate, even for a non-integer number of times.

There is a version of the stable manifold theorem for autonomous vector fields which can be applied directly to (16).

Consider the system $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x)$, with an autonomous vector field $f : \Omega \to \mathbb{R}^N$, and assume that $p_* \in \Omega$ is an equilibrium point. As usual, we denote by Φ^t the time-t map. We define the **linearised vector field** as the autonomous linear vector field $\dfrac{\mathrm{d}X}{\mathrm{d}t} = A \cdot X$ in $\mathbb{R}^N$, with $A = \mathrm{D}f(p_*)$. The reader can check that this definition is consistent with what we did in § 3.4, when linearising the pendulum (passage from system (13) to (15)).

One can prove that $\mathrm{D}\Phi^t(p_*)$ is the time-t map of the vector field A, *i.e.* the linearisation of the flow map is the flow map of the linearised system.[17] Thus

$$\mathrm{D}\Phi^\tau(p_*) = \exp(\tau A).$$

[17] It is easy to check that the family of linear maps $\big(\mathrm{D}\Phi^t(p_*)\big)$ satisfies the one-parameter group property by applying the chain rule to differentiate the relation $\Phi^s \circ \Phi^t = \Phi^{s+t}$. One may then resort to formula (12) to compute the infinitesimal generator, which is easily seen to be a linear vector field. The key is that the components of the differentials $\mathrm{D}\Phi^t(p_*)$ and $\mathrm{D}f(p_*)$ correspond to the action of the operators $\dfrac{\partial}{\partial x_j}$, and these operators commute with the operator $\dfrac{\partial}{\partial t}$ used for the time-derivative (*cf.* the end of § 5.3 and footnote 20 in the appendix).

We say that p_* is a **hyperbolic equilibrium point** for f if the matrix A has no eigenvalues on the imaginary axis. This is equivalent to the hyberbolicity of any of the matrices $\exp(\tau A)$, $\tau \neq 0$, *i.e.* a hyperbolic equilibrium point of a vector field is a hyperbolic fixed point of any of its time-τ maps (except Φ^0 which is the identity map). Moreover, the stable subspace $E^{(s)}$ is the same for all the linear maps $\exp(\tau A)$, $\tau > 0$, and similarly the unstable subspace $E^{(u)}$ is the same for all of them (both spaces are exchanged for $\tau < 0$).

The stable manifold theorem for such a vector field f asserts the existence of submanifolds $\mathcal{W}^{(s)}$ and $\mathcal{W}^{(u)}$ containing p_*, tangent to $E^{(s)}$ and $E^{(u)}$ at p_*, which are the common stable and unstable manifolds of all the maps Φ^τ, $\tau > 0$. These sets are invariant by the flow of f, and

$$\mathcal{W}^{(s)} = \{\, p \in \Omega \mid \Phi^t(p) \xrightarrow[t\to+\infty]{} p_* \,\}, \quad \mathcal{W}^{(u)} = \{\, p \in \Omega \mid \Phi^t(p) \xrightarrow[t\to-\infty]{} p_* \,\}.$$

In the example (16), we had obtained the general solution

$$\gamma(t) = \Phi^t(x,y,z) = \left(x\,\mathrm{e}^{-t},\, y\,\mathrm{e}^{-t} + x^2(\mathrm{e}^{-t} - \mathrm{e}^{-2t}),\, -\tfrac{x^2}{3}\,\mathrm{e}^{-2t} + (z + \tfrac{x^2}{3})\mathrm{e}^{t} \right),$$

with arbitrary initial condition (x,y,z), which shows that imposing $z + \frac{x^2}{3} = 0$ is the only way of preventing the solution from escaping to infinity.

Another easy example is the equilibrium point $p_* = (\pi, 0)$ of the pendulum (13): the curve called Γ^+ in § 3.5 is a piece of the stable manifold, whereas Γ^- is a piece of its unstable manifold. Figure 4 should make it clear what the complete stable and unstable manifolds are: if we consider this system as a vector field in $\mathbb{R}^2$, the stable manifold of p_* is $\{\, (x,v) \in \,]-\pi, 3\pi[\,\times \mathbb{R} \mid v = 2\sqrt{\tfrac{g}{\ell}}\cos(x/2) \,\}$ for instance; observe that we must then consider $p'_* = (-\pi, 0)$ as another hyperbolic fixed point, the unstable manifold of which is $\{\, (x,v) \in \,]-3\pi, \pi[\,\times \mathbb{R} \mid v = 2\sqrt{\tfrac{g}{\ell}}\cos(x/2) \,\}$. In other words, Γ^+ is both a piece of $\mathcal{W}^{(s)}(p_*)$ and of $\mathcal{W}^{(u)}(p'_*)$, this is called a **heteroclinic connection**.

When the same picture is viewed in $\mathbb{S}^1 \times \mathbb{R}$, the fixed points p_* and p'_* are identified, and the manifolds $\mathcal{W}^{(s)}(p_*)$ and $\mathcal{W}^{(u)}(p_*)$ both coincide with $\Gamma^+ \cup \{p_*\} \cup \Gamma^-$. We then say that Γ^+ and Γ^- are **homoclinic** connections.

The coincidence of both manifolds must be regarded as a consequence of the low dimension. In dimension $N \geq 3$, even when there exists a first integral, one can have a hyperbolic equilibrium point of a vector field for which the stable and unstable manifolds do not coincide but there exists a homoclinic connection, *i.e.* an orbit contained in the intersection of these manifolds. The situation can then be much more complicated than in the case of the pendulum (in fact, for a vector field with $N = 3$ without non-trivial first integrals, it can be as complicated as in the case of a discrete system with $d = 2$ like the standard map).

4.6 Let us end by another interesting application of the stable manifold theorem to autonomous vector fields: we now suppose that the vector field $f : \Omega \to \mathbb{R}^N$ admits a periodic orbit γ_*. Consider a point p_* on the periodic orbit and a hypersurface Σ transversal to γ_* at p_*, and let P_Σ be the corresponding Poincaré return map.

For this map, p_* is a fixed point, which may be hyperbolic. It turns out that the hyperbolicity of the fixed point does not depend on the choice of p_* and Σ (because, if we have two choices (p_*, Σ) and (p'_*, Σ'), the maps P_Σ and $P_{\Sigma'}$ are related—the proper word is "conjugate"—by the flow map between Σ and Σ'). We can thus declare that γ_* is **hyperbolic periodic orbit** of f if p_* is a hyperbolic fixed point of P_Σ for a choice of (p_*, Σ).

In this situation, we can apply the stable manifold theorem to P_Σ and we get submanifolds $\mathcal{W}^{(s)}(p_*, P_\Sigma)$ and $\mathcal{W}^{(u)}(p_*, P_\Sigma)$ of Σ, of dimensions $d^{(s)}$ and $d^{(u)}$, with $d^{(s)} + d^{(u)} = \dim \Sigma = N - 1$. From this, one can deduce that

$$\mathcal{W}^{(s)}(\gamma_*) = \{\, p \in \Omega \mid \operatorname{dist}(\Phi^t(p), \gamma_*) \xrightarrow[t \to +\infty]{} 0 \,\},$$
$$\mathcal{W}^{(u)}(\gamma_*) = \{\, p \in \Omega \mid \operatorname{dist}(\Phi^t(p), \gamma_*) \xrightarrow[t \to -\infty]{} 0 \,\}$$

are submanifolds of dimensions $d^{(s)} + 1$ and $d^{(u)} + 1$ containing γ_*.

Again, even with relatively small dimension, things can be complicated. Consider for instance a periodically forced pendulum

$$\left| \begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= v \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{g}{\ell} \sin x + \varepsilon a(t, x), \end{aligned} \right.$$

where $\varepsilon > 0$ is a small parameter, the function a is 2π-periodic in both its arguments and we suppose $a(t, \pi) = 0$ for all t. We can view this as an autonomous vector field in $\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1$ (see the very end of § 3.7), which possesses a hyperbolic 2π-periodic solution $\gamma_* : t \mapsto (\pi, 0, t)$. We can then play the previous game with $\Sigma = \{\, t \equiv 0 \,\}$ (for instance). The upshot is that the periodic orbit has 2-dimensional stable and unstable manifolds in the 3-dimensional phase space, which intersect Σ along the 1-dimensional stable and unstable manifolds of $(\pi, 0, 0)$ for P_Σ; the possible homoclinic connections are illustrated in figure 43 of [Ch], § 4. The 2-dimensional phase portrait of P_Σ may very well look as chaotic as the one of the standard map (figure 1).

5 Appendix: a brief reminder on calculus

The basic object here is a real-valued function defined on an open subset of $\mathbb{R}^N$ and its regularity properties.

5.1 Let Ω be a subset of $\mathbb{R}^N$, where $N \geq 1$ is an integer. The norm of a vector $U = (U_1, \ldots, U_N)$ of $\mathbb{R}^N$ is defined to be the non-negative real number

$$\|U\| = \sqrt{U_1^2 + \cdots + U_N^2}.$$

For any two points $p = (x_1, \ldots, x_N)$ and $p' = (x'_1, \ldots, x'_N)$ in Ω, we denote by $\overrightarrow{pp'}$ the vector $(x'_1 - x_1, \ldots, x'_N - x_N) \in \mathbb{R}^N$ and we set $\mathrm{dist}(p, p') = \|\overrightarrow{pp'}\|$. Conversely, if $p \in \Omega$ and $U \in \mathbb{R}^N$, we define the point $p + U \in \mathbb{R}^N$ as the point of coordinates $(x_1 + U_1, \ldots, x_N + U_N)$, *i.e.* the unique point p' such that $\overrightarrow{pp'} = U$, but this point $p + U$ does not necessarily belong to Ω.

We say that a point p is **interior to** Ω if all the points of the form $p + U$ with $\|U\|$ small enough belong to Ω, *i.e.* if there exists $\rho > 0$ (a number which depends on p and which may be very small) such that $p + U \in \Omega$ as soon as $\|U\| < \rho$.

Consider for instance

$$\mathcal{B} = \{\, p \in \mathbb{R}^N \mid \mathrm{dist}(0, p) \leq 1 \,\}$$

(the closed unit ball centred at the origin). The point $p_0 = (\frac{9}{10}, 0, \ldots, 0)$ belongs to $\mathcal{B}$ and is interior to $\mathcal{B}$, since any point of the form $p_0 + U$ with $\|U\| < \frac{1}{10}$ belongs to $\mathcal{B}$. But a point like $p_1 = (1, 0, \ldots, 0)$, which is also in $\mathcal{B}$, is not interior to $\mathcal{B}$, since for any $\rho > 0$ (even very small) there is at least a vector U with $\|U\| < \rho$ and $p_1 + U \notin \mathcal{B}$ (consider for instance $U = (\rho/2, 0, \ldots, 0)$). It is easy to see that the points of $\mathcal{B}$ which are not interior to $\mathcal{B}$ are precisely those p such that $\mathrm{dist}(0, p) = 1$ (they are called the boundary points).

The set Ω is said to be an **open** subset of $\mathbb{R}^N$ if all the points of Ω are interior to Ω. Thus the closed ball $\mathcal{B}$ is not open[18], whereas the set $\{\, p = (x_1, \ldots, x_N) \mid \mathrm{dist}(0, p) < 1 \,\}$ is open.

5.2 The hypothesis that Ω is open is useful because it means that, for any point $p \in \Omega$, small enough variations $p + U$ of the point also belong to Ω. Thus, if a function $f : \Omega \to \mathbb{R}$ is given, we can evaluate the functions f at these points too and study the differences $f(p + U) - f(p)$ for small enough $\|U\|$.

A function $f : \Omega \to \mathbb{R}$ is said to be **continuous** at a point $p \in \Omega$ if $f(p + U) - f(p)$ tends to 0 as $\|U\| \to 0$. This means that for any $\varepsilon > 0$, small though it may be, one can guarantee $|f(p+U) - f(p)| < \varepsilon$ simply by taking $\|U\|$ sufficiently small.

The function is said to be **differentiable** at p if there exists a linear map $A_p : \mathbb{R}^N \to \mathbb{R}$ such that $f(p + U) - f(p) = A_p \cdot U + b_p(U)$, where the "error

[18] But if you define a **closed** subset of $\mathbb{R}^N$ as the complement of an open subset, you will find that the closed ball $\mathcal{B}$ is... closed!

term" $b_p(U)$ can be neglected in front of U, *i.e.* $b_p(U) = \|U\|\varepsilon_p(U)$ where $\varepsilon_p(U)$ tends to 0 as $\|U\| \to 0$ (typically, the error term can be of the order of $\|U\|^2$).

The general form of a linear map $A : \mathbb{R}^N \to \mathbb{R}$ is $A \cdot U = \alpha_1 U_1 + \cdots + \alpha_N U_N$ with real coefficients $\alpha_1, \ldots, \alpha_N$; in the case of a differentiable function at p, these coefficients are called the **partial derivatives** of f at p and are denoted $\dfrac{\partial f}{\partial x_j}(p)$. The linear map A_p itself is called the **differential** of f at p and is denoted $\mathrm{d}f(p)$. To summarize,

$$f(p+U) = f(p) + \mathrm{d}f(p)\cdot U + \|U\|\varepsilon_p(U),$$
$$\mathrm{d}f(p)\cdot U = \frac{\partial f}{\partial x_1}(p)U_1 + \cdots + \frac{\partial f}{\partial x_N}(p)U_N, \qquad \varepsilon_p(U) \xrightarrow[\|U\|\to 0]{} 0. \quad (17)$$

One should remember that this formula corresponds to a linear approximation of the variations of f around p.

Clearly, formula (17) implies that $f(p+U) - f(p)$ tends to 0 as $\|U\| \to 0$. Hence, a function which is differentiable at a point is necessarily continuous at this point. Here are examples with $N = 1$ and $\Omega = \mathbb{R}$.

– The function $f : x \mapsto x^2$ is differentiable at each point of $\mathbb{R}$, with $\mathrm{d}f(x)\cdot U = 2xU$. (Indeed: the variation around x is $(x+U)^2 - x^2 = 2xU + U^2$, thus the linear approximation is $2xU$, while the error term is quadratic in this example.)

– The function $g : x \mapsto |x|$ (absolute value) is continuous at each point of $\mathbb{R}$, differentiable at each $x > 0$ with $\mathrm{d}g(x)\cdot U = U$, differentiable at each $x < 0$ with $\mathrm{d}g(x)\cdot U = -U$, but not differentiable at 0.

– The function $h : x \mapsto \frac{x}{|x|}$, whatever its definition at 0, is not continuous at 0 (it is the "sign function": $h(x) = +1$ if $x > 0$, $h(x) = -1$ if $x < 0$).

In $\mathbb{R}^N$, the function $f_1 : p = (x_1, \ldots, x_N) \mapsto \mathrm{dist}(0,p) = \sqrt{x_1^2 + \cdots + x_N^2}$ is continuous at every point, and differentiable at each point of $\mathbb{R}^N \setminus \{0\}$, but it is not differentiable at the origin itself. The function $f_2 : (x_1, \ldots, x_N) \mapsto x_1^2 + \cdots + x_N^2$ is differentiable at every point of $\mathbb{R}^N$. Their differentials at a point $p = (x_1, \ldots, x_N)$ are related with the scalar product:

$$\mathrm{d}f_1(p)\cdot U = \frac{x_1U_1 + \cdots + x_NU_N}{\sqrt{x_1^2 + \cdots + x_N^2}}, \quad \mathrm{d}f_2(p)\cdot U = 2(x_1U_1 + \cdots + x_NU_N), \qquad U \in \mathbb{R}^N.$$

The differentials in these examples can be computed quite simply by using the tools indicated in the following (beginning with f_2 and then writing $(f_1)^2 = f_2$).

5.3 We say that a function f is continuous on an open subset Ω of $\mathbb{R}^N$ if it is continuous at each point of Ω. We say that it is differentiable on Ω if it is differentiable at each point of Ω.

In the last case, at each $p \in \Omega$ we get N real numbers $\dfrac{\partial f}{\partial x_1}(p), \ldots, \dfrac{\partial f}{\partial x_N}(p)$, so

we can define N functions[19]

$$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N} : \Omega \to \mathbb{R}$$

called the partial derivatives of f. If these functions are continuous on Ω, f is said to be **continuously differentiable** on Ω (or of C^1 class).[20]

For functions of a single variable x, *i.e.* when $N = 1$ and Ω is an open subset of $\mathbb{R}$, for instance an open interval $]a, b[$, there is only one possible partial derivative, which is denoted $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ instead of $\dfrac{\partial f}{\partial x}$ and is called simply "the derivative". The derivative of $f :]a, b[\to \mathbb{R}$ is also denoted f'.

For example, if $k \in \mathbb{N}$, the function $f : x \mapsto x^k$ is differentiable[21] on $\mathbb{R}$, with derivative $\dfrac{\mathrm{d}f}{\mathrm{d}x} : x \mapsto k\, x^{k-1}$. The derivative $f' = \dfrac{\mathrm{d}f}{\mathrm{d}x}$ is itself differentiable, with derivative $f'' : x \mapsto k(k-1)\, x^{k-2}$, and in fact f is of C^∞ class.

The exponential function $\exp : x \mapsto \mathrm{e}^x$ is differentiable on $\mathbb{R}$ and has the remarkable property of coinciding with its derivative: $(\exp)' = \exp$ (it is thus of C^∞ class). This function $\exp : \mathbb{R} \to]0, +\infty[$ can be defined[22] as the reciprocal of the logarithm function $\log :]0, +\infty[\to \mathbb{R}$, which is the unique differentiable function on $]0, +\infty[$ which vanishes at the point 1 and which has derivative $(\log)' : x \mapsto 1/x$.

For functions of N variables, the name "partial derivatives" comes from the fact that $\dfrac{\partial f}{\partial x_j}$ can be computed as the derivative of the "partial" function $x_j \mapsto f(x_1, \dots, x_j, \dots, x_N)$, where $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N$ are considered to be fixed, while x_j is the only variable,.

5.4 Linear combinations (with constant coefficients) of differentiable functions are differentiable, with

$$\frac{\partial}{\partial x_j}(\lambda f + \mu g) = \lambda \frac{\partial f}{\partial x_j} + \mu \frac{\partial g}{\partial x_j}$$

for instance.

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[19] Observe that we can also define a map $\mathrm{d}f : p \mapsto \mathrm{d}f(p)$, but it is not a real-valued function, it rather takes its values in the space of all linear maps from $\mathbb{R}^N$ to $\mathbb{R}$.

[20] It often happens that the partial derivatives $\frac{\partial f}{\partial x_j}$ are themselves continuously differentiable; if so, we say that f is of C^2 class, and we can associate with it N^2 second-order partial derivatives $\frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right)$, which are simply denoted $\frac{\partial^2 f}{\partial x_k \partial x_j}$. Classical theorems then tell us that $\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k}$ (Schwarz formula), and that formula (17) can be refined, giving rise to the Taylor formula at second order. Functions of $C^3, C^4, \dots$ class are defined similarly: the functions which are indefinitely differentiable are said to be of C^∞ class.

[21] If k is a negative integer, say $k = -\ell$, $\ell \geq 1$, by definition $x^{-\ell} = 1/x^\ell$ and we get a differentiable function on $\mathbb{R} \setminus \{0\}$, with derivative $-\ell\, x^{-\ell-1}$. If k is a real number, not necessarily integer, x^k is still defined for $x > 0$ (as $\mathrm{e}^{k \log x}$) and we get a differentiable function on $]0, +\infty[$, with derivative $k\, x^{k-1}$.

[22] The exponential function can also be defined as the sum of a series: $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.

As a consequence, the derivative of a polynomial is easily computed. For example $P(x_1, x_2) = \alpha_{0,0} + \alpha_{1,0}x_1 + \alpha_{0,1}x_2 + \alpha_{2,0}x_1^2 + \alpha_{1,1}x_1x_2 + \alpha_{0,2}x_2^2$ implies $\dfrac{\partial P}{\partial x_1}(x_1, x_2) = \alpha_{1,0} + 2\alpha_{2,0}x_1 + \alpha_{1,1}x_2$.

The product of two differentiable functions is a differentiable function, the partial derivatives of which are computed according to **Leibniz rule**:

$$\frac{\partial}{\partial x_j}(fg) = \frac{\partial f}{\partial x_j}g + f\frac{\partial g}{\partial x_j}.$$

5.5 We can also deal with vector-valued functions. Let $M \geq 1$ be integer. Considering a function $F : \Omega \to \mathbb{R}^M$ amounts to considering simultaneously M functions $F_1, \ldots, F_M : \Omega \to \mathbb{R}$, which are the components of F, *i.e.*

$$F(p) = \big(F_1(p), \ldots, F_M(p)\big) \in \mathbb{R}^M, \qquad p \in \Omega \subset \mathbb{R}^N.$$

All the previous definitions can be extended to this case. For instance F is said to be differentiable if each component F_i is differentiable; we then group together the partial derivatives of the components to obtain vector-valued functions

$$\frac{\partial F}{\partial x_j} = \left(\frac{\partial F_1}{\partial x_j}, \ldots, \frac{\partial F_M}{\partial x_j}\right) : \Omega \to \mathbb{R}^M, \qquad j = 1, \ldots, N.$$

Thus, for fixed j, the operator $\dfrac{\partial}{\partial x_j}$ sends a vector-valued function $F : \Omega \to \mathbb{R}^M$ to a vector-valued function $\dfrac{\partial F}{\partial x_j} : \Omega \to \mathbb{R}^M$.

In this situation, it is worth grouping together the differentials of the components at a given point p: formula (17) indeed yields M equations

$$F_i(p + U) = F_i(p) + \mathrm{d}F_i(p) \cdot U + \|U\|\varepsilon_{p,i}(U)$$

which can be compactly written as

$$F(p + U) = F(p) + \mathrm{D}F(p) \cdot U + \|U\|\varepsilon_p(U), \tag{18}$$

where $\varepsilon_p(U)$ is now a vector-valued function of U which tends to 0 as $\|U\| \to 0$, and

$$\mathrm{D}F(p) \cdot U = \big(\mathrm{d}F_1(p) \cdot U, \ldots, \mathrm{d}F_M(p) \cdot U\big) \in \mathbb{R}^M, \qquad U \in \mathbb{R}^N.$$

We thus define a linear map $\mathrm{D}F(p) : \mathbb{R}^N \to \mathbb{R}^M$, which is called the **tangent map** to F at p (some authors call it the derivative of F at p). As in § 1.5 we can associate with it a matrix[23], the elements of which are the coefficients $\frac{\partial F_i}{\partial x_j}(p)$.

[23]Be careful that the variable is now $U = (U_1, \ldots, U_N)$ (rather than p which is considered to be fixed in the previous formulas). Moreover, contrarily to what happened in § 1.5, the matrix of $\mathrm{D}F(p)$ is not necessarily square since we did not assume $M = N$.

We emphasize that at each point p we have a linear map $\mathrm{D}F(p)$ from $\mathbb{R}^N$ to $\mathbb{R}^M$, thus the function $\mathrm{D}F$ is a rather sophisticated object; this mixture of analysis and linear algebra sometimes causes troubles to students.

5.6 In practice, an important formula is the so-called **chain rule**, which tells us how to deal with the composition of two differentiable functions. Suppose for instance that Ω is an open subset of $\mathbb{R}^N$, Ω' is an open subset of $\mathbb{R}^M$, $F = (F_1, \dots, F_M)$ is a differentiable vector-valued function on Ω, and φ is a differentiable real-valued function on Ω'. Suppose furthermore that F takes its values in Ω', so that we can consider the composed function $\varphi \circ F$:

$$\begin{array}{ccccc} \Omega & \xrightarrow{F} & \Omega' & \xrightarrow{\varphi} & \mathbb{R} \\ p & \longrightarrow & F(p) & \longrightarrow & \varphi \circ F(p). \end{array}$$

Then the function $\varphi \circ F$ is differentiable on Ω and

$$\mathrm{d}(\varphi \circ F)(p) = \mathrm{d}\varphi(F(p)) \circ \mathrm{D}F(p),$$

which means that

$$\mathrm{d}(\varphi \circ F)(p) \cdot U = \mathrm{d}\varphi(F(p)) \cdot [\mathrm{D}F(p) \cdot U], \qquad U \in \mathbb{R}^N,$$

or, equivalently,

$$\frac{\partial(\varphi \circ F)}{\partial x_j}(p) = \frac{\partial \varphi}{\partial y_1}(F(p))\frac{\partial F_1}{\partial x_j}(p) + \cdots + \frac{\partial \varphi}{\partial y_M}(F(p))\frac{\partial F_M}{\partial x_j}(p)$$

if we call $(y_1, \dots, y_M)$ the coordinates in Ω'.

At this point, we refer to textbooks for the fundamental theorem of analysis, which is the Implicit Function Theorem.

5.7 Of particular interest for differential equations is the case when $N = 1$ and $M \geq 1$: a vector-valued function of a single real variable can be interpreted as a **parametrised curve** in $\mathbb{R}^M$.

As an open subset of $\mathbb{R}$ on which the function is defined, we choose an open interval $I =]a, b[$, and we consider a differentiable function

$$\gamma = (\gamma_1, \dots, \gamma_M) : I \to \mathbb{R}^M.$$

Instead of x_1 or x, the variable in I will be denoted t. The only possible partial derivative (since there is only one variable) is denoted $\dfrac{\mathrm{d}\gamma}{\mathrm{d}t}$ (instead of $\dfrac{\partial\gamma}{\partial t}$) and is called "the derivative" (as in the case of real-valued functions of a single variable—*cf.* § 4.3); it is a vector-valued function

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \left(\frac{\mathrm{d}\gamma_1}{\mathrm{d}t}, \dots, \frac{\mathrm{d}\gamma_M}{\mathrm{d}t}\right) : I \to \mathbb{R}^M.$$

One also uses the notation $\gamma' = (\gamma'_1, \dots, \gamma'_M)$ to denote the same function (or the notation $\dot\gamma$). Sometimes, one writes $\frac{\mathrm{d}}{\mathrm{d}t}\big(\gamma(t)\big)_{|t=t_0}$ instead of $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t_0)$ or $\gamma'(t_0)$.

If we interpret the variable t as time, we may consider γ as the time-parametrisation of a curve, and we may speak of $\gamma(t)$ as of a **mobile point** in $\mathbb{R}^M$. For that reason $\gamma'(t)$ is called the **velocity vector** at time t. The time-derivative of the velocity is nothing but the acceleration $\gamma'' = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}\gamma}{\mathrm{d}t}\right)$ (when it exists, *e.g.* when γ is of C^2 class). Observe that with these notations, formula (17) reads

$$\gamma_i(t+u) = \gamma_i(t) + \gamma'_i(t)u + u\,\varepsilon_{t,i}(u), \qquad i = 1, \dots, M,$$

with $\varepsilon_{t,i}(u) \xrightarrow[u\to 0]{} 0$, or equivalently $\gamma'(t) = \dfrac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = \lim\limits_{u\in\mathbb{R},\, u\to 0} \dfrac{\gamma(t+u)-\gamma(t)}{u}$.

An example with $M = 2$ is the parametrisation of a circle of radius R with constant angular velocity ω: $\gamma(t) = (R\cos(\omega t), R\sin(\omega t))$, for which the velocity vector $\gamma'(t) = (-\omega R\sin(\omega t), \omega R\cos(\omega t))$ has constant norm $|\omega|R$.

5.8 Let us end with two useful consequences of the chain rule for a parametrised curve $\gamma : I \to \mathbb{R}^M$.

a) A differentiable function $\theta : J \to I$ (where J is an open interval in $\mathbb{R}$) can be used as a change of parametrisation, *i.e.* we may consider the differentiable function $s \mapsto \Gamma(s) = \gamma\big(\theta(s)\big)$. Then $\dfrac{\mathrm{d}}{\mathrm{d}s}\big[\gamma_i\big(\theta(s)\big)\big]_{|s=s_0} = \dfrac{\mathrm{d}\gamma_i}{\mathrm{d}t}\big(\theta(s_0)\big)\,\dfrac{\mathrm{d}\theta}{\mathrm{d}s}(s_0)$ for each i, which yields

$$\Gamma'(s_0) = \frac{\mathrm{d}}{\mathrm{d}s}\big[\gamma\big(\theta(s)\big)\big]_{|s=s_0} = \theta'(s_0)\,\gamma'\big(\theta(s_0)\big). \tag{19}$$

A common abuse of notation in this situation consists in using the same letter γ to denote $\Gamma(s)$ and $\gamma(t)$, and the same letter t for the old variable and for the change of variable θ, which yields the formula

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}s}\,\frac{\mathrm{d}\gamma}{\mathrm{d}t}$$

(which may be easier to remember). Here, $\dfrac{\mathrm{d}t}{\mathrm{d}s}$ must be understood as the function $\theta' = \dfrac{\mathrm{d}\theta}{\mathrm{d}s}$.

As an example of application of formula (19), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\big[\gamma(\lambda s)\big] = \lambda\,\gamma'(\lambda s), \quad \frac{\mathrm{d}}{\mathrm{d}s}\big[\gamma(a+s)\big] = \gamma'(a+s). \tag{20}$$

for arbitrary constants λ and a.

b) If a differentiable function g is given in $\mathbb{R}^M$ (with values in $\mathbb{R}$)—in physics, this is sometimes called an "observable"—, by composition we get a real-valued function $t \mapsto g\big(\gamma(t)\big)$ (which may be interpreted as the result of a measurement

which varies with time, because the point at which g is measured is a mobile point). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[g\big(\gamma(t)\big)\big]_{|t=t_0} = \mathrm{d}g\big(\gamma(t_0)\big)\cdot\gamma'(t_0). \tag{21}$$

In the right-hand side, the linear map $\mathrm{d}g\big(\gamma(t_0)\big)$ is evaluated on the vector $\gamma'(t_0)\in\mathbb{R}^M$, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[g\big(\gamma(t)\big)\big]_{|t=t_0} = \frac{\partial g}{\partial x_1}\big(\gamma(t_0)\big)\gamma_1'(t_0) + \cdots + \frac{\partial g}{\partial x_M}\big(\gamma(t_0)\big)\gamma_M'(t_0).$$

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