Phase resetting, noise, and synchrony

Bard Ermentrout
Supported by NSF,NIMH

Paris June 2005
Tip o’ the hat to

- Nathan Urban, Roberto Galan, Nicolas Fourcaud
- David Saunders
- Tay Netoff, John White
- Boris Gutkin & Alex Reyes
Phase resetting

Experimental computations

Maps and noise

Invariant densities etc

Stochastic synchrony
Biological rhythms are governed by nonlinear oscillators.

The Phase-resetting curve (PRC) is defined as

\[ \Delta(\phi) = 1 - \frac{P'(\phi)}{P}, \quad \phi \equiv \frac{s}{P} \]

tells us how an oscillator is changed due to the timing of inputs.
Well known in circadian literature

- Human Bright light (Czeisler, 2003)
- Arabidopsis
- Drosophila
- cyanobacteria (Xu et al)
...and in other areas

![Graphs showing phase resetting and stimulus time](image1.png)

- **Pteroptx cribellata**
- **Pteroptx malaccae**
- **Luciola pupilla**

**Stimulus time (msec)**

**Phase resetting**

**Yeast metabolism – ammonium sulfite**

(Murray et al. 2003)
1. Shea-Brown, Izhikevich, GBE

2. Hopf gives sinusoidal PRC that is positive and negative

\[ PRC(x) = K \sin x + \phi \]

3. SNIC is strictly positive

\[ PRC(x) = K (1 - \cos x) \]

4. Ionic currents also alter (cf Kopell, Gutkin, Stieffel)
Hints for neurons

RSU cortex (Reyes & Fetz)

Mitral cell (Galen et al)

No M-current (SNIC)

M-current (Hopf)
Why do we care about the PRC?

- Uses for entrainment are well known (Glass, Belair, ad infinitum)
- Coupling is less known, but
  - Shape of PRC determines phase patterns
  - Synchrony vs other patterns (Goel, Kreso, NK, White)
Aside: Averaging and all that

1. General coupled oscillators:

\[
\frac{dX_j}{dt} = F(X_j) + \sum_k G_{jk}(X_j, X_k)
\]

2. Reduce to phase model:

\[
\frac{d\theta_j}{dt} = \omega_j + \sum_k p_{jk}(\theta_k, \theta_j) \Delta_j(\theta_j)
\]

3. Average, if \( p_{jk} \) small:

\[
\frac{d\theta_j}{dt} = \omega_j + \sum_k H_{jk}(\theta_k - \theta_j)
\]

\[
H_{jk}(\phi) = \frac{1}{T} \int_0^T \Delta(t) p_{jk}(t + \phi) \, dt
\]
Winfree model

\[ p_{jk}(\theta_k, \theta_j) = c_{jk} P(\theta_k) \]

\[ \frac{d\theta_j}{dt} = \sum_k c_{jk} P(\theta_k) \Delta(\theta_j) \]

Assume

- \[ \sum_k c_{jk} = c, \; c_{jk} \geq 0 \]
- \[ \phi' = 1 + cP(\phi)\Delta(\phi) \] has soln, \( \phi(0) = \phi(T) + 1 \).
- \[ \int_0^T P(\phi(t)) \Delta'(\phi(t)) \, dt < 0 \]

Then, \( \theta_j(t) = \phi(t) \) is asymptotically stable.
Beyond synchrony - end aside

Arrays of NN coupled oscillators

\[ \sin(x) \quad \sin(x) + b \cos(x) \]

100 sine oscillators on sphere

\( \alpha - \text{EEG: rotating waves} \)

scalp EEG – Viktor Jirsa
Real PRCs are noisy!

Entorhinal cortex

Galen et al (2005)

Netoff et al., Figure 2
J. Neurophys 2004
The obvious method is not terribly useful for neurons due to the noise.

There are several other methods:
- PRC from PSTH
- PRC from assumed function
  - Galan et al method
  - Izhikevich method
\[ P \int_{0}^{t} \text{PSTH}(s) \, ds = F^{-1}(t), \quad F(t) = t + \Delta(t) \]

where \( P \) is the period and \( \Delta \) is the PRC.
\[ s' = s - P \Delta (P - s) = F(s) \]

\[
\int_0^t \text{PSTH}(t') dt' = \Pr\{s' < t\} \\
= \Pr\{F(s) < t\} \\
= \Pr\{s < F^{-1}(t)\} \\
= \frac{F^{-1}(t)}{P}
\]
Least squares method

\[ \theta' = \omega + \sum_i \delta(t - t_i) \Delta(\theta) \]

\[ \Delta(\theta) = \sum_{n=0}^{M} a_n \cos n\theta + b_n \sin n\theta \]

Times of inputs known, times of neural spikes known, between spikes, \( \theta = \omega t \), so unknowns \( a_n, b_n \) are linear functions of data!

\[ 2\pi = \omega T_j + \sum_{n=0}^{m} a_n \cos n\omega T_j + b_n \sin n\omega T_j \]

where \( T_j \) are interspike intervals.
Application to data

Morris Lecar model

Class I

Class II

Mitral Cell

Galen et al. PRL 2005

Phase resetting, noise, and synchrony – p.18/38
Given the PRC, now what?

Couple them into networks:

\[ \theta'_j = \omega_j + \sum_k J_{jk} \delta(\theta_k) \Delta_j(\theta_j) \]

or more generally

\[ \theta'_j = \omega_j + \sum_{k,i} J_{jk}(t - t^i_k) \Delta_j(\theta_j) \]

where \( J_{jk}(t) \) is the signal of cell \( k \) to cell \( j \) and \( t^i_k \) are the firings.

Winfree model:

\[ \theta_j = \omega_j + \sum_k J_{jk} P_k(\theta_k) \Delta_j(\theta_j) \]
Two oscillators

Assume each cell has frequency, $\omega_{A,B}$.

$$\phi'_A \rightarrow \frac{\omega_A}{\omega_B} [1 - F_B(\phi_B)]$$
$$\phi'_B \rightarrow \frac{\omega_B}{\omega_A} [1 - F_A(\phi_A)].$$

where $F(x) = x + \Delta(x)$.

Suppose they are identical – then phase-difference comes from iterates of the (noisy) maps

$$x \rightarrow 1 - x - \Delta(x).$$
Assume the noisy map has the form:

$$X_{n+1} = 1 - X_n - \Delta(X_n) + Z_n R(X_n)$$

$$\equiv F(X_n) + Z_n R(X_n)$$

where $R(x)$ is the phase-dependent noise.

Let

$$P_n(x)dx = \Pr\{x < X_n < x + dx\}$$

Then

$$P_n(x) = \int_0^1 Q \left( \frac{x - F(y)}{R(y)} \right) \frac{P_n(y)}{R(y)} dy.$$
Experimental phase-densities

Dynamic clamp/Monte carlo

Netoff et al J.Neurophy '05
Stationary solutions

A: Stationary distribution

B: Inhibitory Coupling

C: mean PRC

D: Derived maps \( x = 1 - F(x) \)
Phase-dependent noise

Excitatory

Inhibitory

Phase resetting, noise, and synchrony – p.24/38
Mixed cells

- Excitatory leads
- With faster I cells, phase-drift
- Faster E gives sharper locking
Bistable media

Expt’l maps seem to have either synchrony or antiphase as solutions

What if the deterministic map is bistable?

\[ \Delta(x) = a \sin x + 0.25(1 + \cos x) - 0.35 \sin 2x \]
Transition rates

In noisy bistable systems, want jumping rate

Let \( I \subset [0, 1] \) and \( X_n \) sequence from noisy map:

\[
N(x) = \inf \{ n \mid X_n \notin I, \; X_0 = x \}.
\]

\( g_1(x) = E[N(x)] \) satisfies:

\[
g_1(x) = 1 + \int_0^1 \frac{Q[(y - G(x))/R(x)]}{R(x)} g_1(y) \chi_I(y) \, dy
\]

is the mean “first passage time.”

Similar eqns for higher moments
Invariant density \((a=0)\)

- \(P(x)\)
- \(\sigma\)
- \(x(0)\)

Mean first passage

- \(E[N]\)
- \(\sigma\)

\(g_1(x)\)

- \(E[N]\)
- \(x(0)\)

Synchrony

- \(x(0)\)

Antiphase

- \(x(0)\)
Stochastic synchrony

1. Local field potential oscillations (20-40 Hz) in olfactory bulb
2. Mediated by inhibitory granule cells
3. IPSPs last 300-500 msec - too long for 20 Hz
4. Small random 10-20 msec IPSPs on top
1. Two oscillators share common noise
2. Not enough to effect firing rate
3. Push the phases together
Mitral cell responses

Simulation

Phase resetting, noise, and synchrony – p.31/38
Power boost

CC for different input correlations

Spectral properties of inputs

Power gain

Inputs

common ipscs

Phase resetting, noise, and synchrony – p.32/38
Could it happen *in vivo*

Data from Nathan Schoppa

Synchronous transient events

Events are GABA–mediated

Data from Nathan Schoppa

Phase resetting, noise, and synchrony – p.33/38
Correlated GC IPSPs

Data from Nathan Schoppa

Precise shared IPSPs

Cumulative histogram

Phase resetting, noise, and synchrony – p.34/38
Two oscillators with shared Poisson inputs

\[
x' = 1 + \sum_k \delta(t - t_k) \Delta(x)
\]
\[
y' = 1 + \sum_k \delta(t - t_k) \Delta(y)
\]

This becomes a random map:

\[
x_{n+1} = T_n + x_n + \Delta(x_n)
\]
\[
y_{n+1} = T_n + y_n + \Delta(y_n)
\]

Difference:

\[
z_{n+1} = F'(v_n)z_n \quad F(x) = x + \Delta(x)
\]
Theory: Poisson, ctd

Contraction:

\[ \rho = \lim_{n \to \infty} \left( \prod_{j=1}^{n} |F'(v_n)| \right)^{\frac{1}{n}} \]

\[ \lambda \equiv \log \rho = \int_{0}^{1} P(v) \log|F'(v)| \, dv \]

But \( P(v) \) is uniform and if \( F(x) \) is monotone:

\[ \lambda = \int_{0}^{1} \log |F'(v)| \, dv \leq \log \left( \int_{0}^{1} F'(v) \, dv \right) = 0 \]

from Jensens inequality

Equality only when trivial PRC, so always contracts!
1. Ritt, PRE, Teramae PRL 2004, GBE (back of envelope)

2. Two uncoupled oscillators w/ white noise

\[
\begin{align*}
    dx &= (1 + a)dt + (pdW + (1 - p)dW_x)\Delta(x) \\
    dy &= dt + (pdW + (1 - p)dW_y)\Delta(y)
\end{align*}
\]

3. Consider the difference:

\[
\begin{align*}
    dz &= a\, dt + p\Delta'(u)zdW + (1 - p)[\Delta(y)dW_y - \Delta(x)dW_x] \\
    &\approx a\, dt + pDzdW + (1 - p)GdW_1
\end{align*}
\]

where \( D = \|\Delta'(u)\|_2 \) and \( G = \|\Delta(u)\|_2 \).

4. Stationary phase distribution:

\[
P(z) = Ke^{a\arctan \frac{dz}{g}} \frac{1}{d^2 z^2 + g^2} \quad d = pD, \quad g = (1 - p)G.
\]

5. Centered around \( a/2d^2 \) so it is strongly synchronizing.
Further questions

- Advantage of explicit maps over Monte carlo
- Analytic expressions for random maps (yes if $\Delta(x)$ is small)
- More than 2 oscillators?
- Infinitely many - Kuramoto analogue?