Phase resetting, noise, and synchrony

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Tip o' the hat to

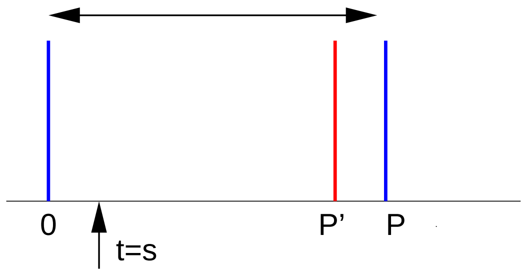
- Nathan Urban, Roberto Galan, Nicolas Fourcaud
- David Saunders
- Tay Netoff, John White
- Boris Gutkin & Alex Reyes

Overview

- Phase resetting
- Experimental computations
- Maps and noise
- Invariant densities etc
- Stochastic synchrony

Phase resetting

- Biological rhythms are governed by nonlinear oscillators



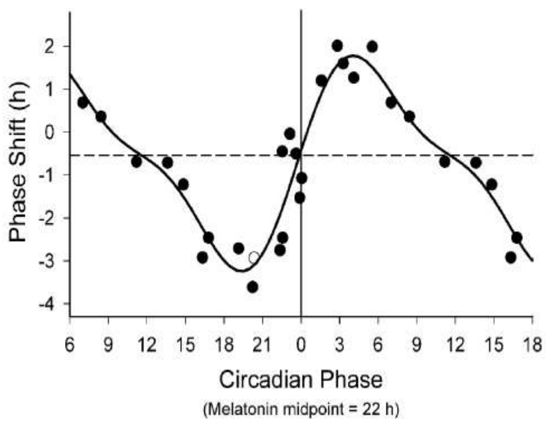
- The *Phase-resetting curve* (PRC) is defined as

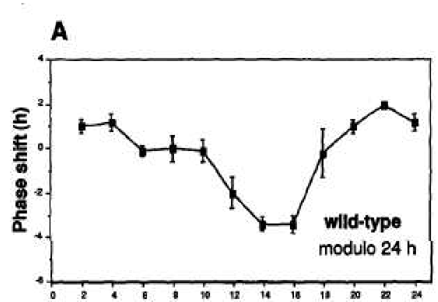
$$\Delta(\phi) = 1 - \frac{P'(\phi)}{P}, \qquad \phi \equiv \frac{s}{P}$$

- tells us how an oscillator is changed due to the timing of inputs.

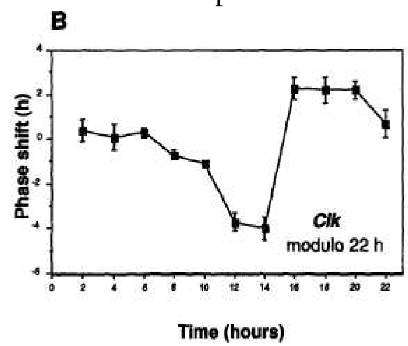
Well known in circadian literature

Human Bright light (Czeisler, 2003)

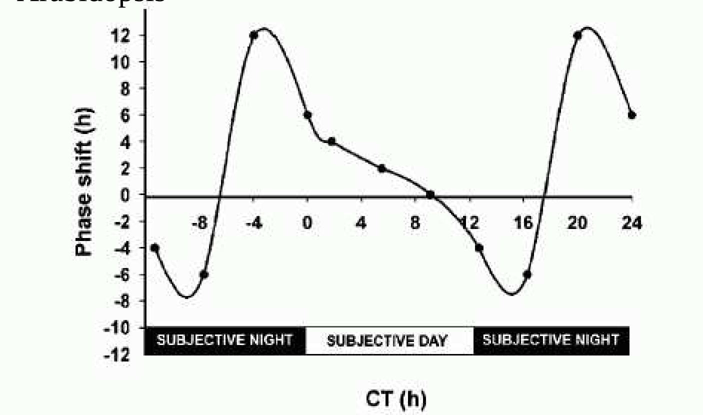


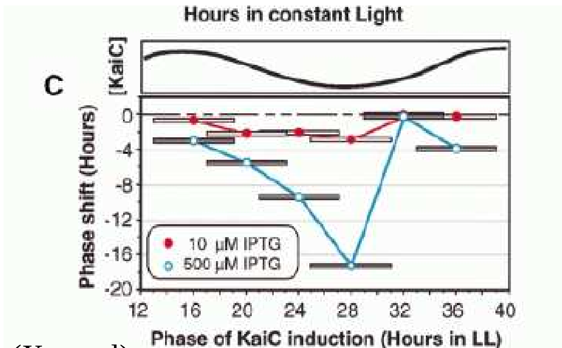


Drosophila



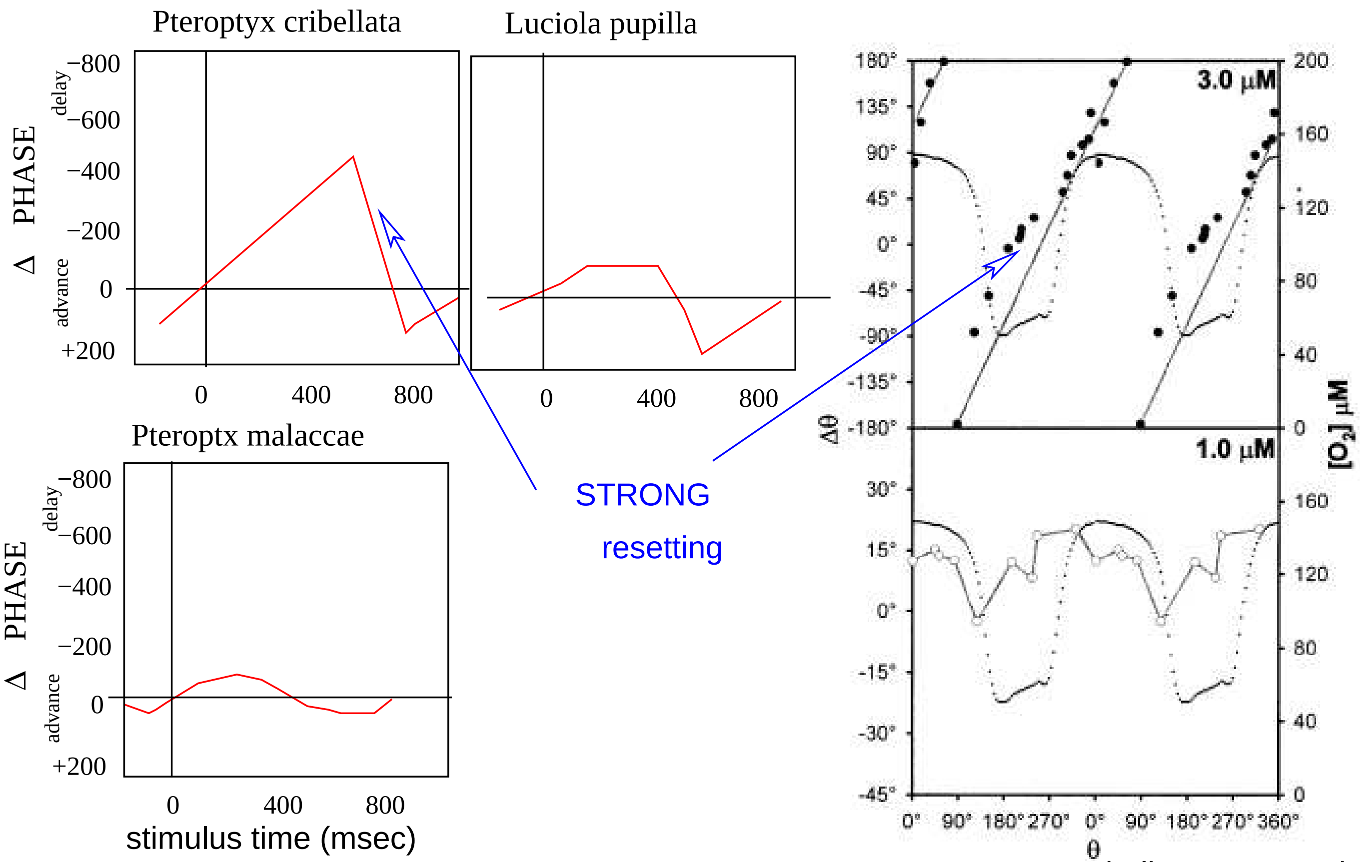
Arabidopsis





cyanobacteria (Xu et al)

. . . and in other areas



Yeast metabolism − ammonium sulfite

(Murray etal 2003)

The PRC depends on bifurcation

1. Shea-Brown, Izhikevich, GBE
2. Hopf gives sinusoidal PRC that is positive and negative

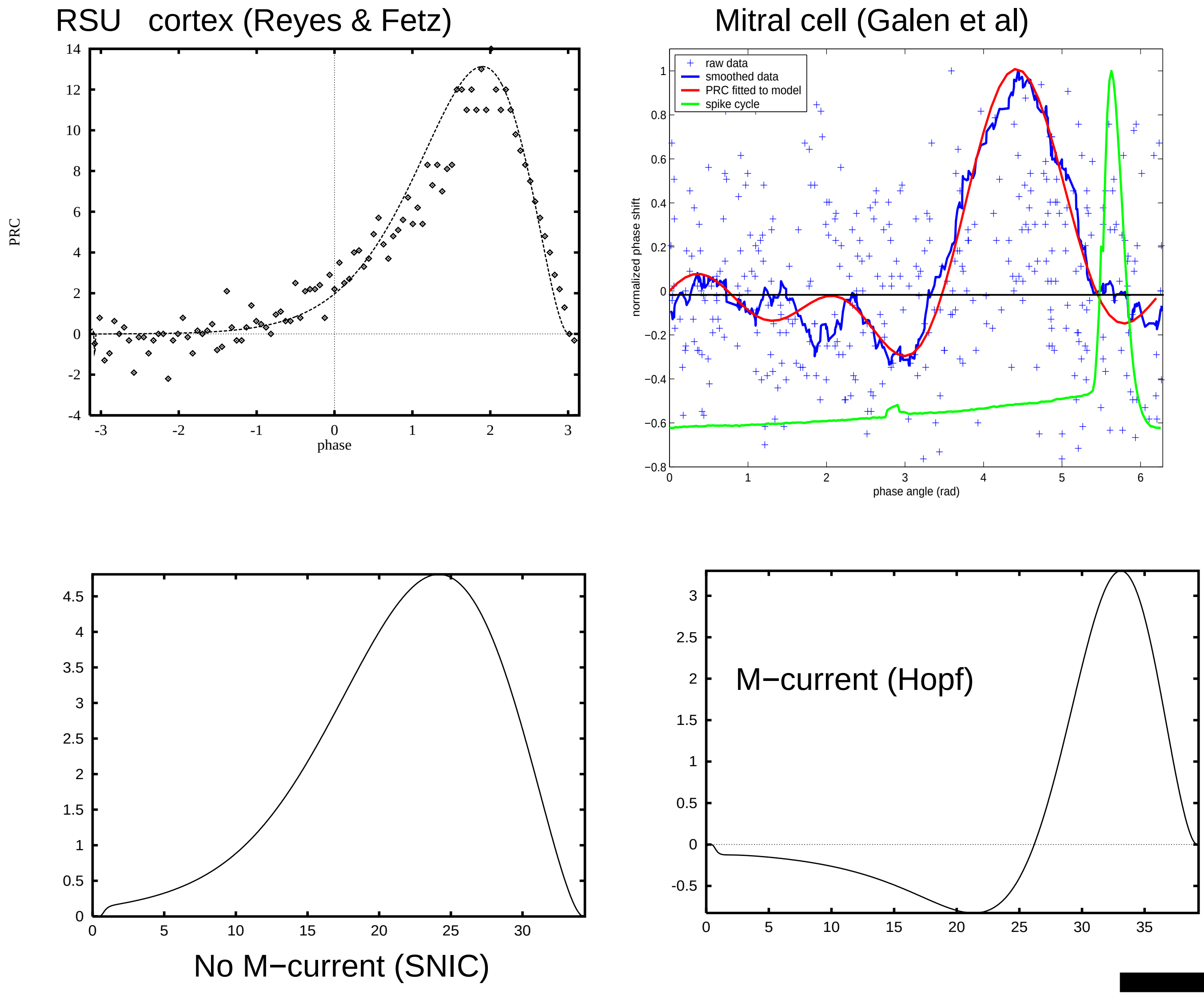
$$\text{PRC}(x) = K \sin x + \phi$$

3. SNIC is strictly positive

$$\text{PRC}(x) = K(1 - \cos x)$$

4. Ionic currents also alter (cf Kopell, Gutkin, Stieffel)

Hints for neurons



Why do we care about the PRC?

- Uses for entrainment are well known (Glass, Belair, ad infinitum)
- Coupling is less known, but
  - Shape of PRC determines phase patterns
  - Synchrony vs other patterns (Goel, Kreso, NK, White)

Aside: Averaging and all that

1. General coupled oscillators:

$$\frac{dX_J}{dt} = F(X_j) + \sum_k G_{jk}(X_j, X_k)$$

2. Reduce to phase model:

$$\frac{d\theta_j}{dt} = \omega_j + \sum_k p_{jk}(\theta_k, \theta_j)\Delta_j(\theta_j)$$

3. Average, if p_{jk} small:

$$\begin{aligned} \frac{d\theta_j}{dt} &= \omega_j + \sum_k H_{jk}(\theta_k - \theta_j) \\ H_{jk}(\phi) &= \frac{1}{T}\int_0^T \Delta(t) p_{jk}(t+\phi)\, dt \end{aligned}$$

Winfree model

$p_{jk}(\theta_k, \theta_j) = c_{jk} P(\theta_k)$

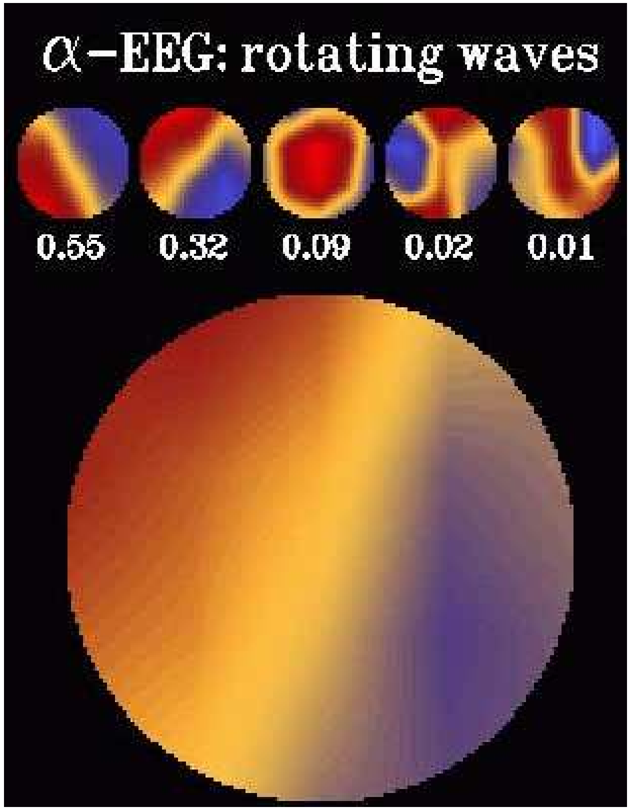
$$\frac{d\theta_j}{dt} = \sum_k c_{jk} P(\theta_k) \Delta(\theta_j)$$

Assume

- $\sum_k c_{jk} = c$, $c_{jk} \geq 0$
- $\phi' = 1 + cP(\phi)\Delta(\phi)$ has soln, $\phi(0) = \phi(T) + 1$.
- $$\int_0^T P(\phi(t))\Delta'(\phi(t))\, dt < 0$$

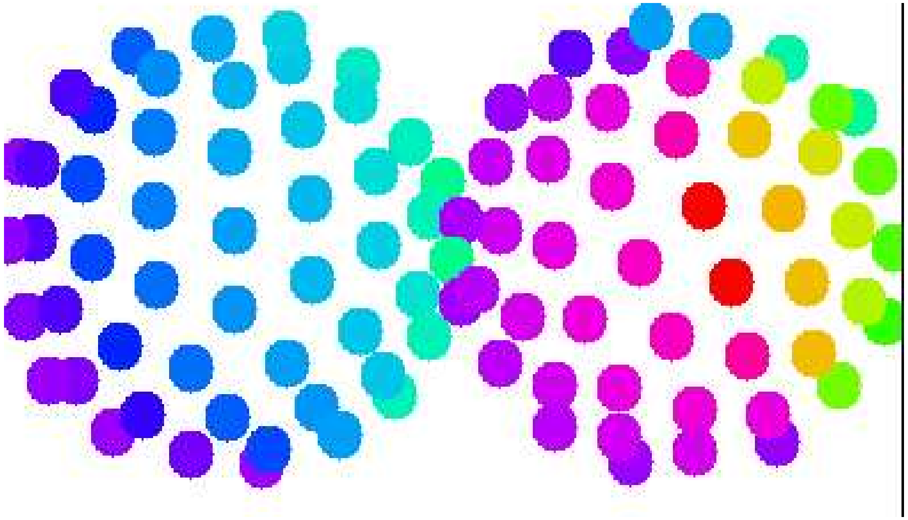
Then, $\theta_j(t) = \phi(t)$ is asymptotically stable.

Beyond synchrony -end aside



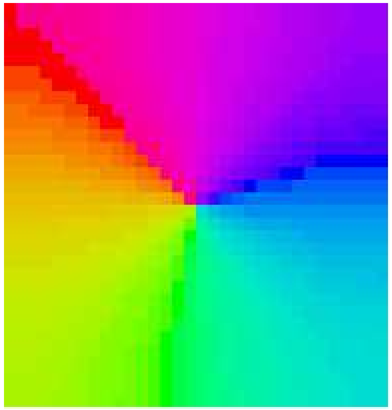
scalp EEG − Viktor Jirsa

100 sine oscillators on sphere



Arrays of NN coupled oscillators

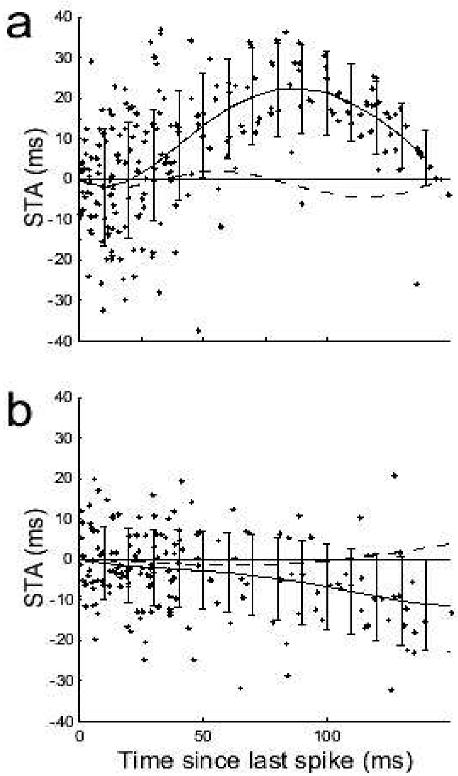
sin(x)



sin(x) + b cos(x)

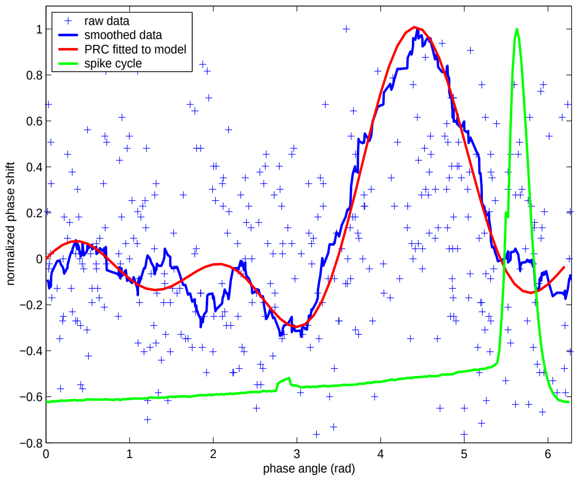
Real PRCs are noisy!

Entorhinal cortex



Netoff et al., Figure 2
J.Neurophys 2004

Mitral cell

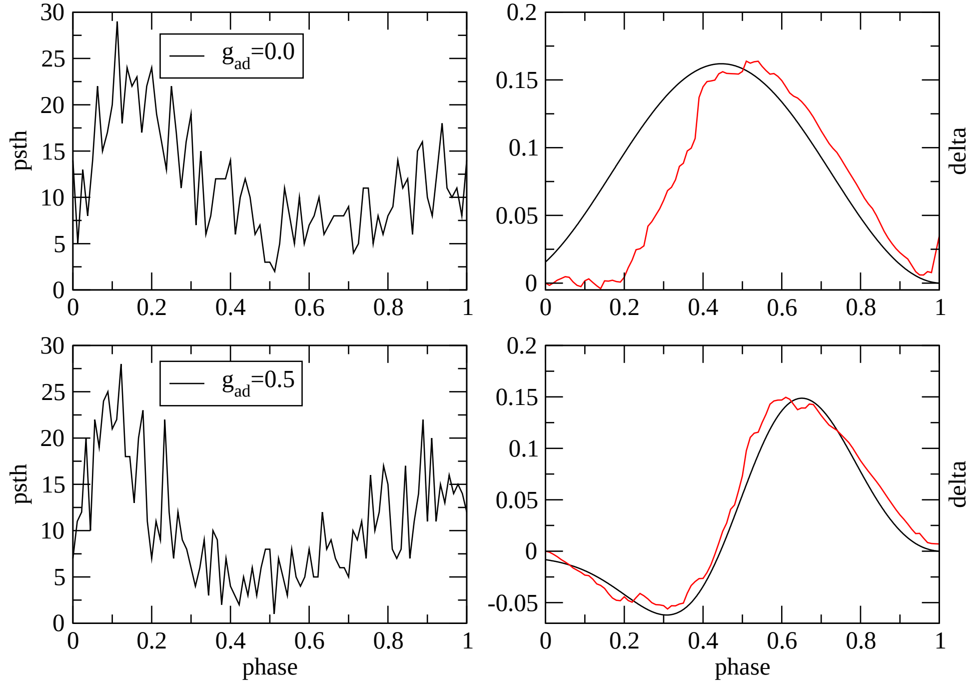


Galen et al (2005)

How do we compute them?

- The obvious method is not terribly useful for neurons due to the noise.
- There are several other methods
  - PRC from PSTH
  - PRC from assumed function
    - Galan et al method
    - Izhikevich method

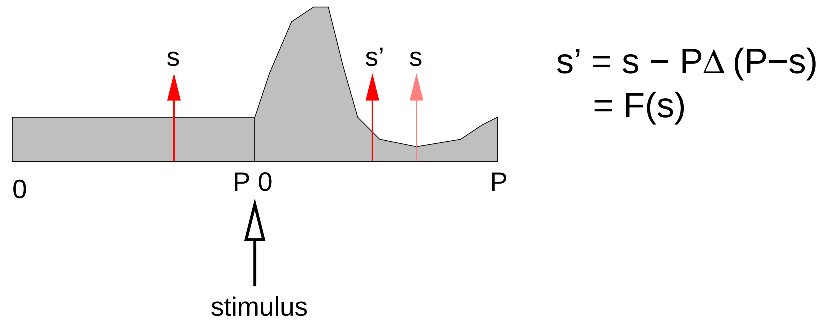
PRC & PSTH



$$P\int_0^t \text{PSTH}(s)\, ds = F^{-1}(t), \quad F(t) = t + \Delta(t)$$

where P is the period and Δ is the PRC.

PRC from PSTH - proof



$$\int_0^t \mathrm{PSTH}(t')dt' = \mathrm{Pr}\{s' < t\}$$
$$= \mathrm{Pr}\{F(s) < t\}$$
$$= \mathrm{Pr}\{s < F^{-1}(t)\}$$
$$= \frac{F^{-1}(t)}{P}$$

Least squares method

$$\theta' = \omega + \sum_i \delta(t - t_i)\Delta(\theta)$$

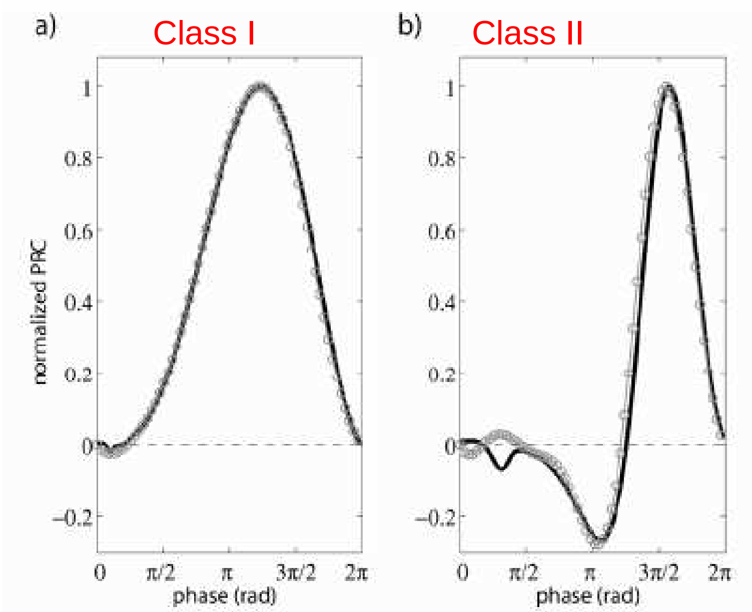
$$\Delta(\theta) = \sum_{n=0}^{M} a_n \cos n\theta + b_n \sin n\theta$$

Times of inputs known, times of neural spikes known, between spikes, $\theta = \omega t$, so unknowns a_n, b_n are **linear** functions of data!

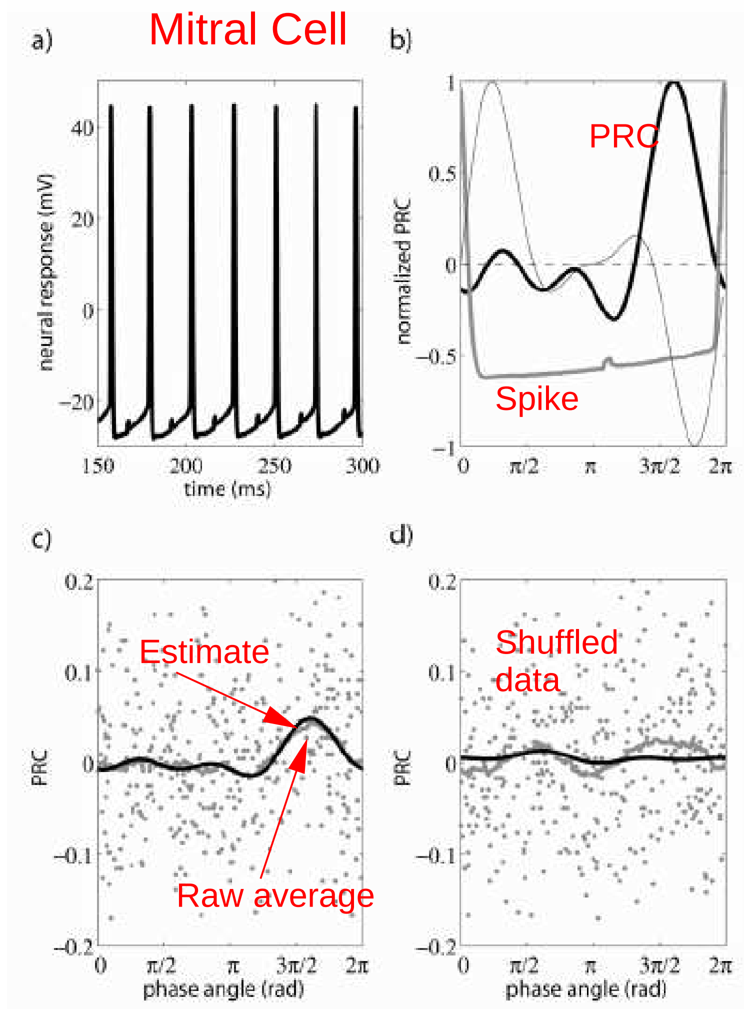
$$2\pi = \omega T_j + \sum_{n=0}^{m} a_n \cos n\omega T_j + b_n \sin n\omega T_j$$

where T_j are interspike intervals.

Application to data



Morris Lecar model



Galen et al PRL 2005

Given the PRC, now what?

- Couple them into networks:

$$\theta'_j = \omega_j + \sum_k J_{jk}\delta(\theta_k)\Delta_j(\theta_j)$$

- or more generally

$$\theta'_j = \omega_j + \sum_{k,i} J_{jk}(t - t^i_k)\Delta_j(\theta_j)$$

where $J_{jk}(t)$ is the signal of cell k to cell j and t^i_k are the fi rings.

- Winfree model:

$$\theta_j = \omega_j + \sum_k J_{jk}P_k(\theta_k)\Delta_j(\theta_j)$$

Two oscillators

- Assume each cell has frequency, $\omega_{A,B}$.

$$
\begin{aligned}
\phi'_A &\longrightarrow \frac{\omega_A}{\omega_B}[1 - F_B(\phi_B)] \\
\phi'_B &\longrightarrow \frac{\omega_B}{\omega_A}[1 - F_A(\phi_A)].
\end{aligned}
$$

  where $F(x) = x + \Delta(x)$.

- Suppose they are identical – then phase-difference comes from iterates of the (noisy) maps

$$x \longrightarrow 1 - x - \Delta(x).$$

Noisy maps

- Assume the noisy map has the form:

$$
\begin{aligned}
X_{n+1} &= 1 - X_n - \Delta(X_n) + Z_n R(X_n) \\
&\equiv F(X_n) + Z_n R(X_n)
\end{aligned}
$$

where $R(x)$ is the phase-dependent noise.

- Let

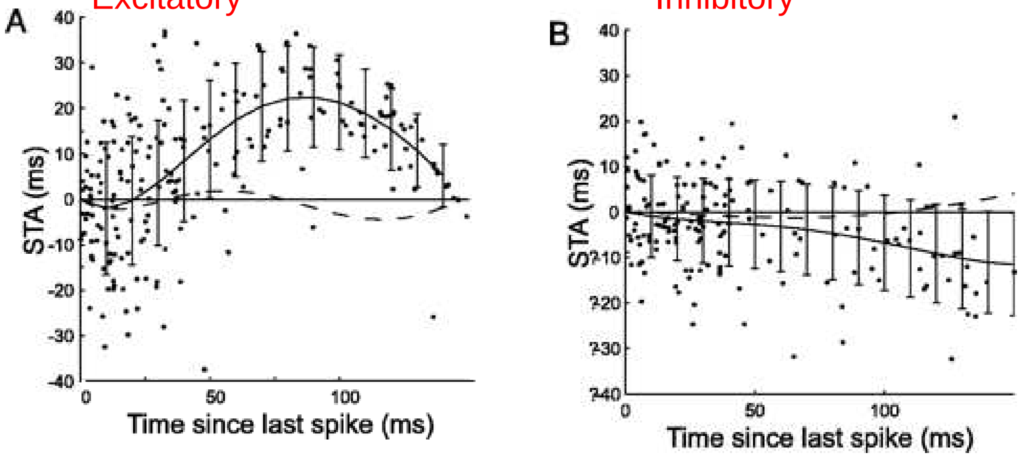
$$P_n(x)dx = \text{Pr}\{x < X_n < x + dx\}$$

- Then

$$P_n(x) = \int_0^1 Q\left(\frac{x - F(y)}{R(y)}\right) \frac{P_n(y)}{R(y)}\, dy.$$

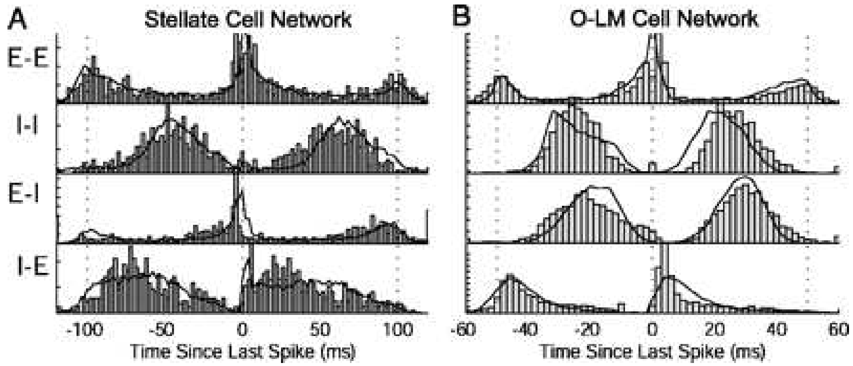
Experimental phase-densities

Excitatory Inhibitory

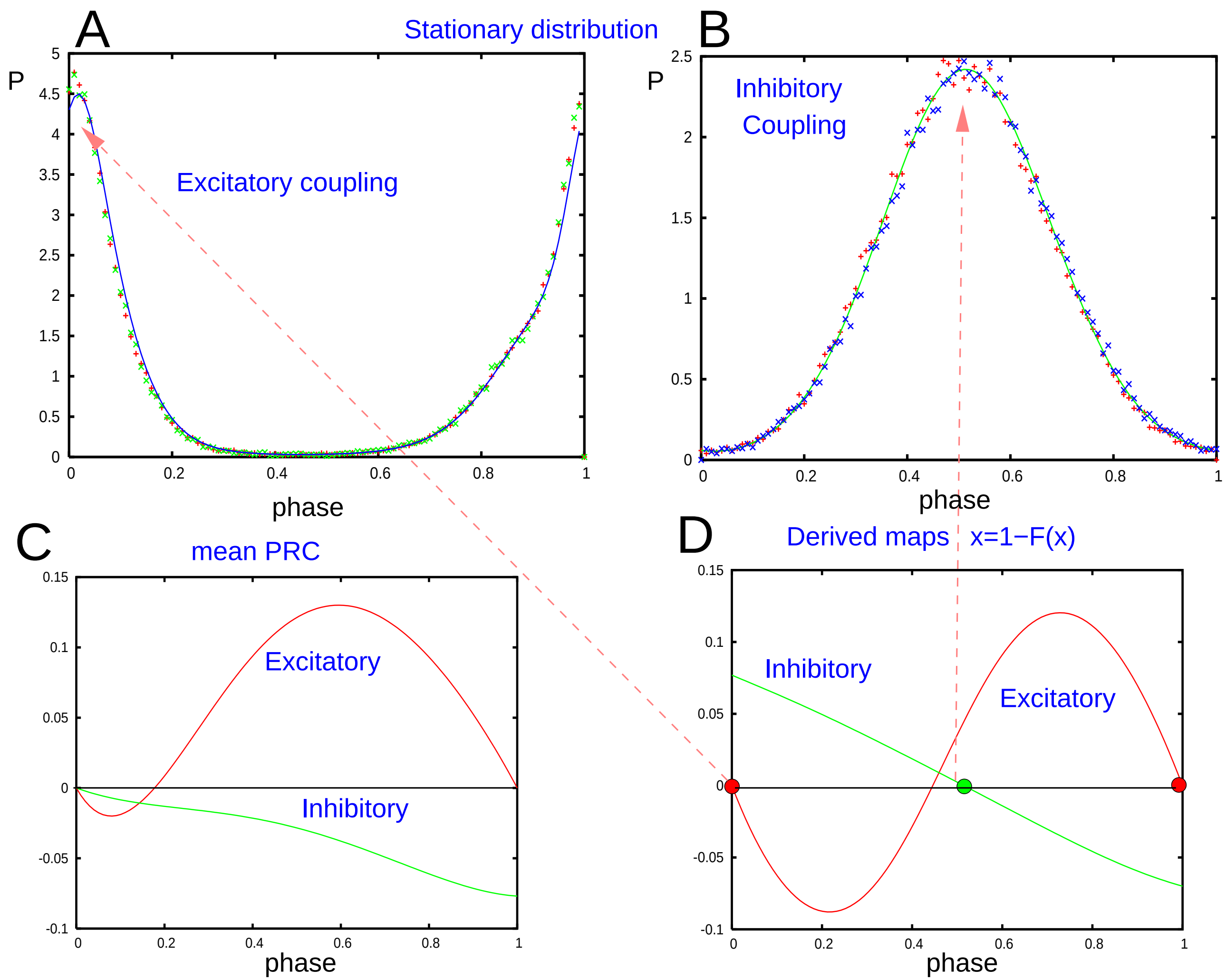


Dynamic clamp/Monte carlo

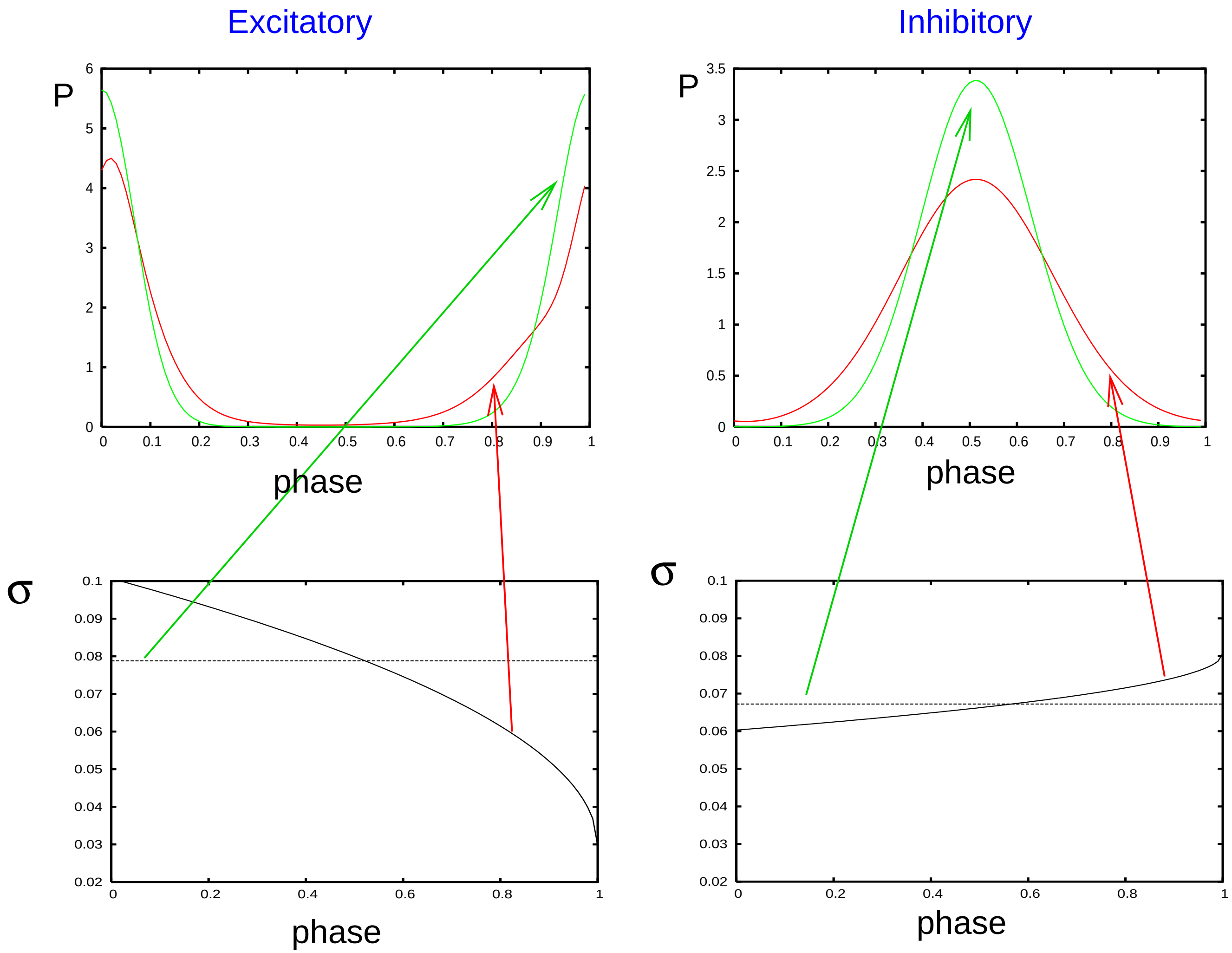
Netoff et al J.Neurophy ’05



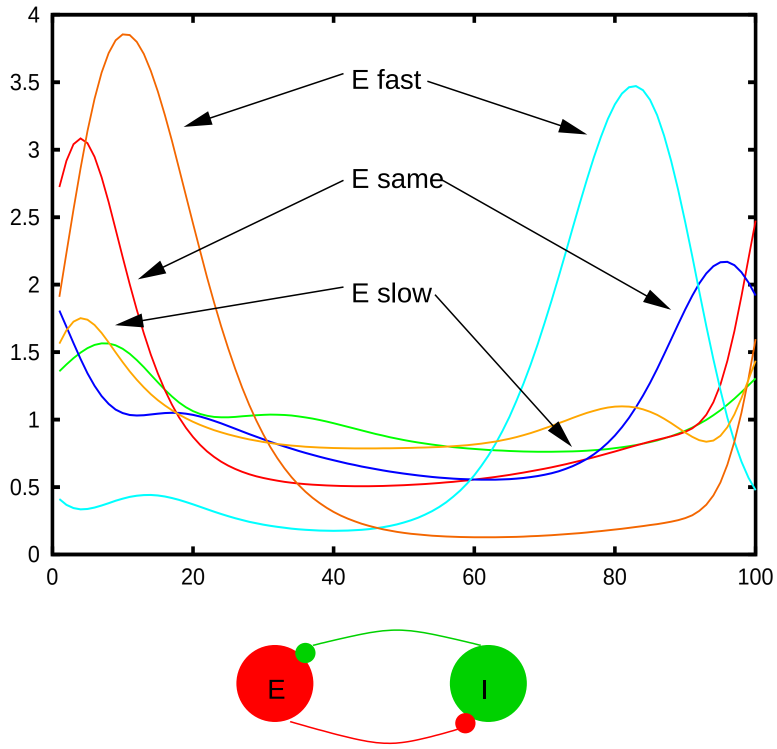
Stationary solutions



Phase-dependent noise



Mixed cells

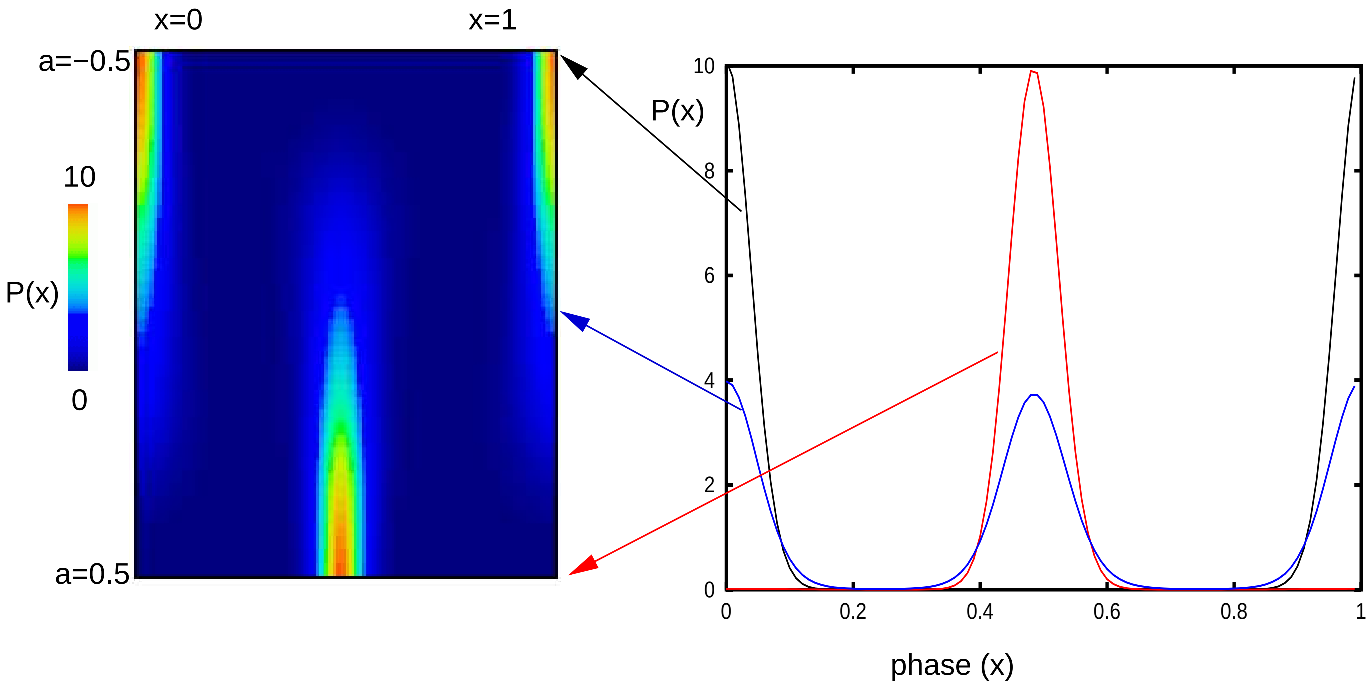


- Excitatory leads
- With faster I cells, phase-drift
- Faster E gives sharper locking

Bistable media

- Expt'l maps seem to have either synchrony or antiphase as solutions
- What if the deterministic map is bistable?

$$\Delta(x) = a \sin x + 0.25(1 + \cos x) - 0.35 \sin 2x$$



Transition rates

- In noisy bistable systems, want jumping rate
- Let $I \subset [0,1]$ and X_n sequence from noisy map:

$$N(x) = \inf\{n \mid X_n \notin I,\ X_0 = x\}.$$

- $g_1(x) = E[N(x)]$ satisfies:

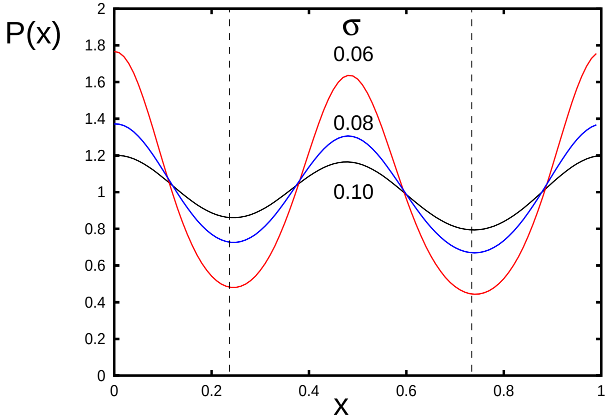
$$g_1(x) = 1 + \int_0^1 \frac{Q[(y - G(x))/R(x)]}{R(x)} g_1(y)\chi_I(y)\, dy$$

is the mean "first passage time."

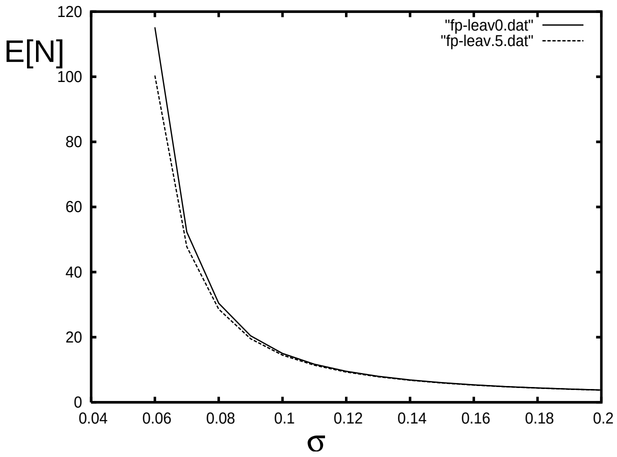
- Similar eqns for higher moments

Example

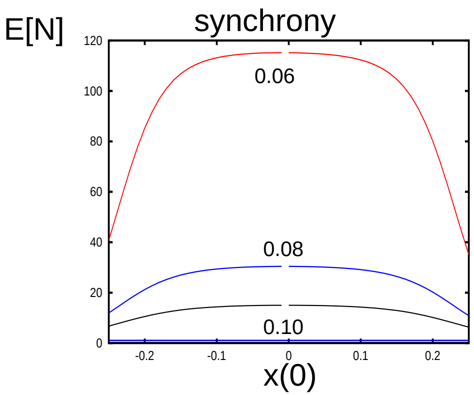
Invariant density (a=0)

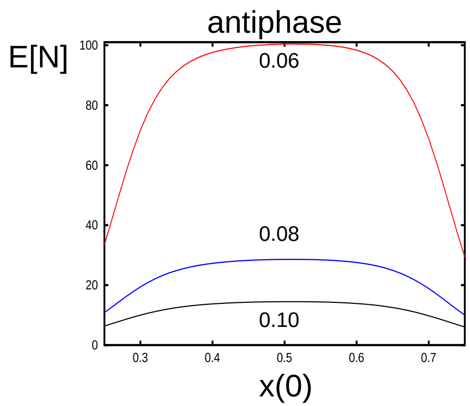


mean first passage



$g_1(x)$





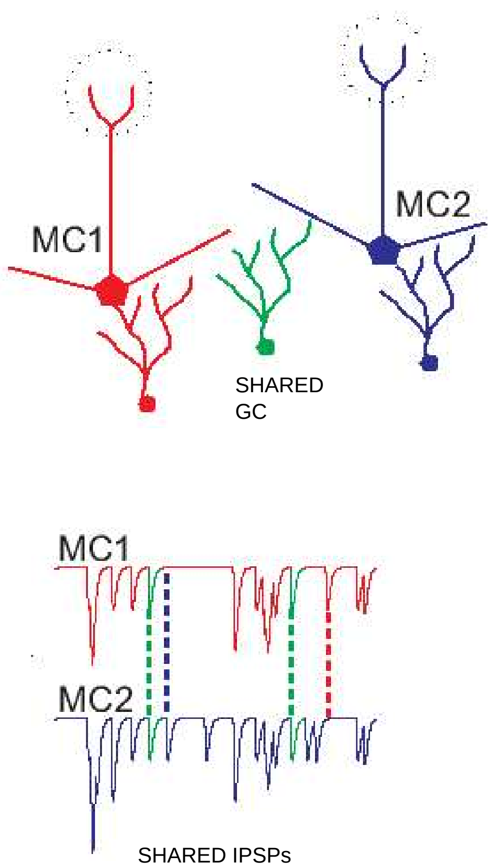
Stochastic synchrony

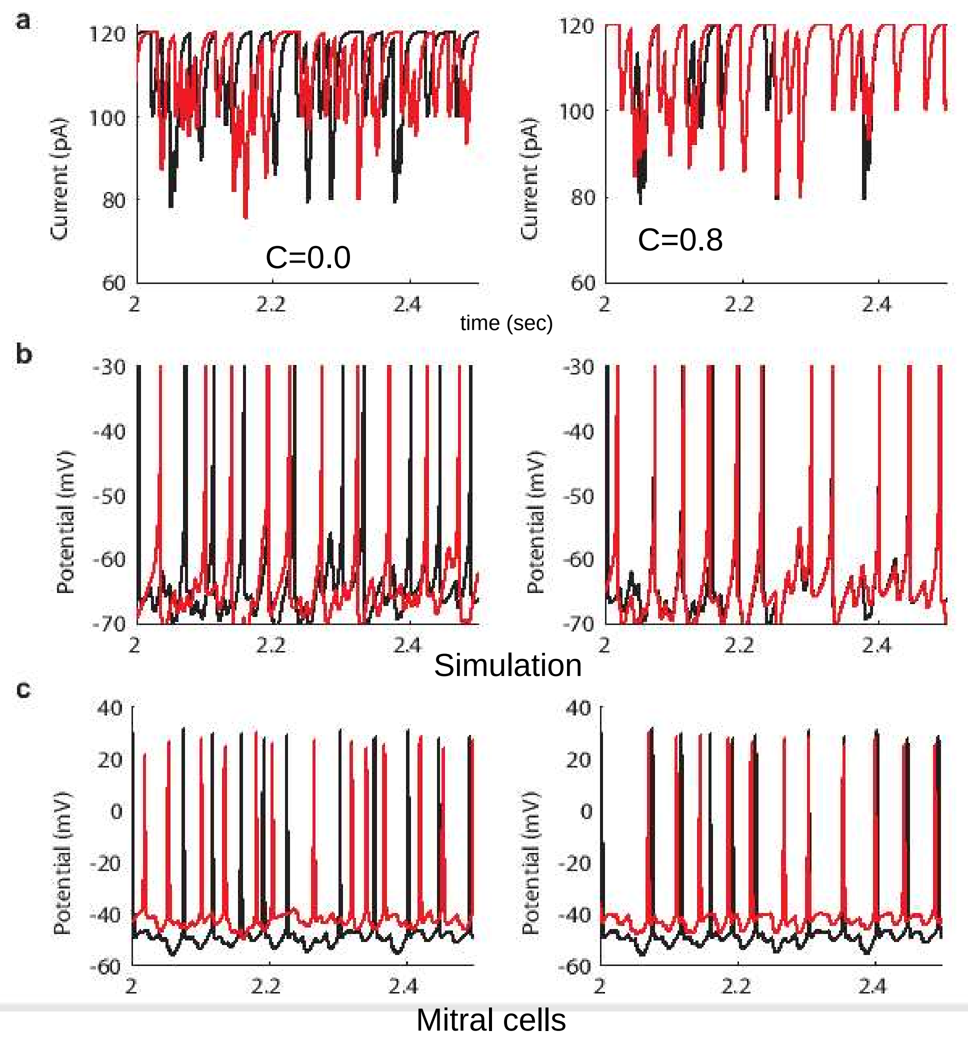
1. Local field potential oscillations (20-40 Hz) in olfactory bulb
2. Mediated by inhibitory granule cells
3. IPSPs last 300-500 msec - too long for 20 Hz
4. Small random 10-20 msec IPSPs on top

Basic idea

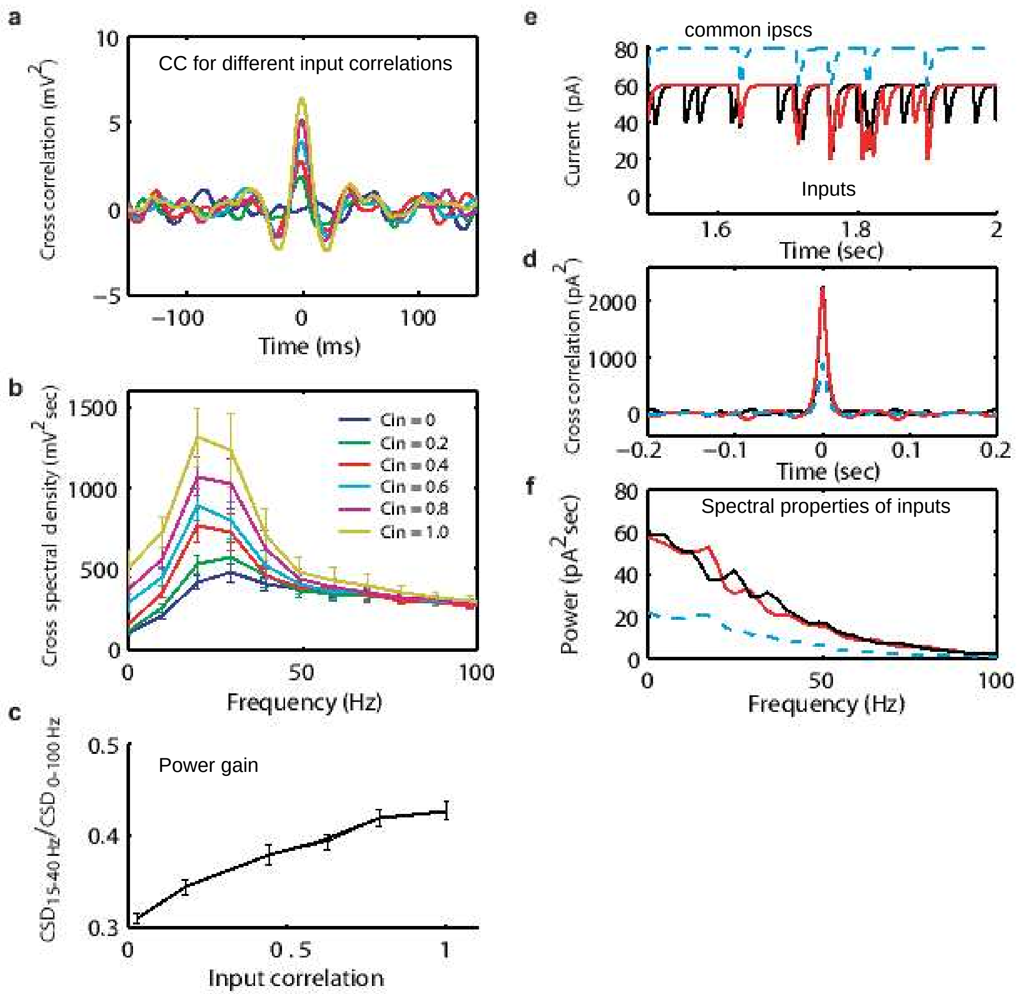
1. Two oscillators share common noise
2. Not enough to effect firing rate
3. Push the phases together

Mitral cell responses



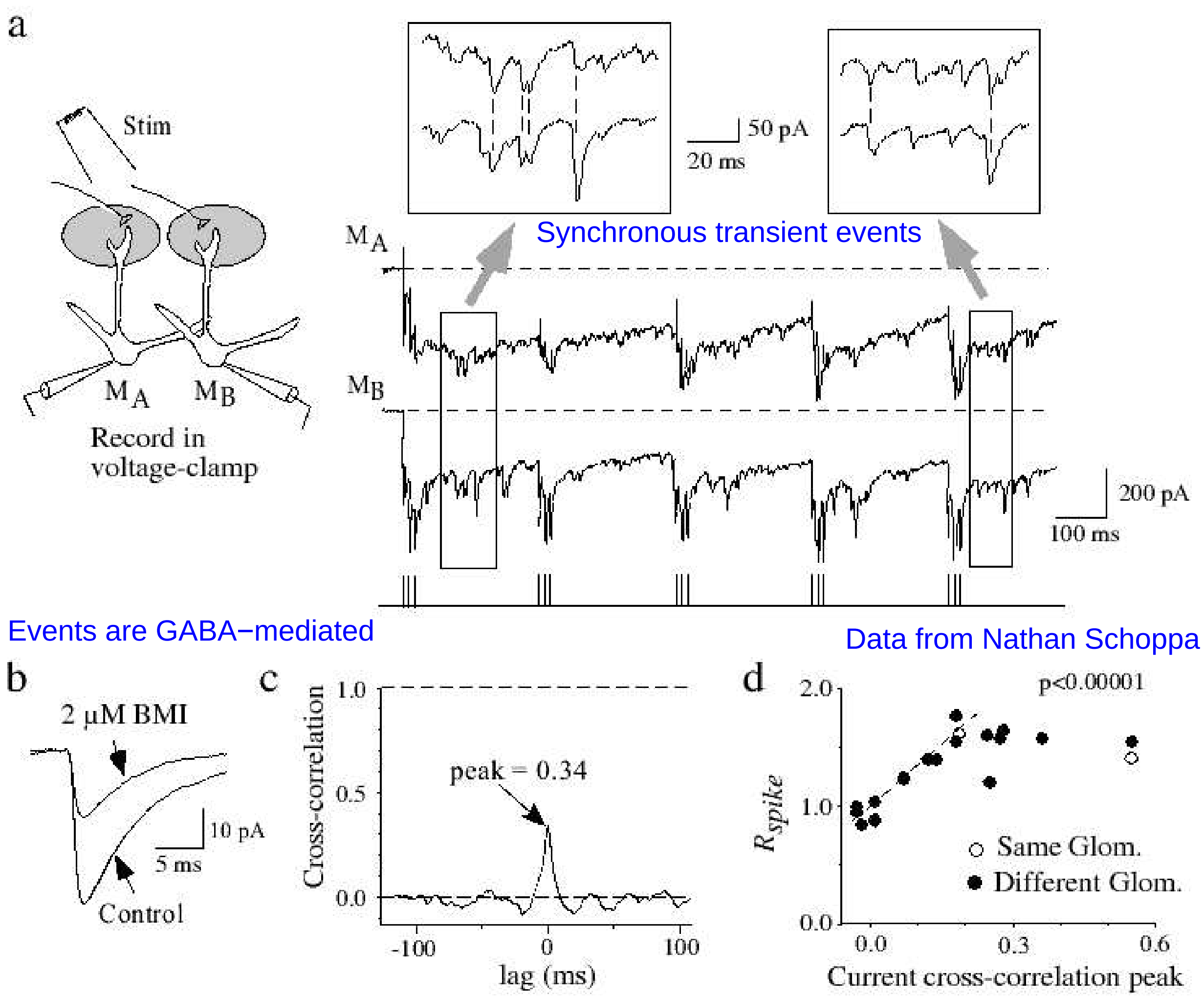


Power boost



a
CC for different input correlations
Cross correlation (mV^2)
Time (ms)
b
Cross spectral density (mV^{2}sec)
Cin = 0
Cin = 0.2
Cin = 0.4
Cin = 0.6
Cin = 0.8
Cin = 1.0
Frequency (Hz)
c
Power gain
CSD$_{15\text{-}40\ Hz}$/CSD$_{0\text{-}100\ Hz}$
Input correlation
e
common ipscs
Inputs
Current (pA)
Time (sec)
d
Cross correlation (pA^2)
Time (sec)
f
Spectral properties of inputs
Power (pA^{2}sec)
Frequency (Hz)

Could it happen *in vivo*



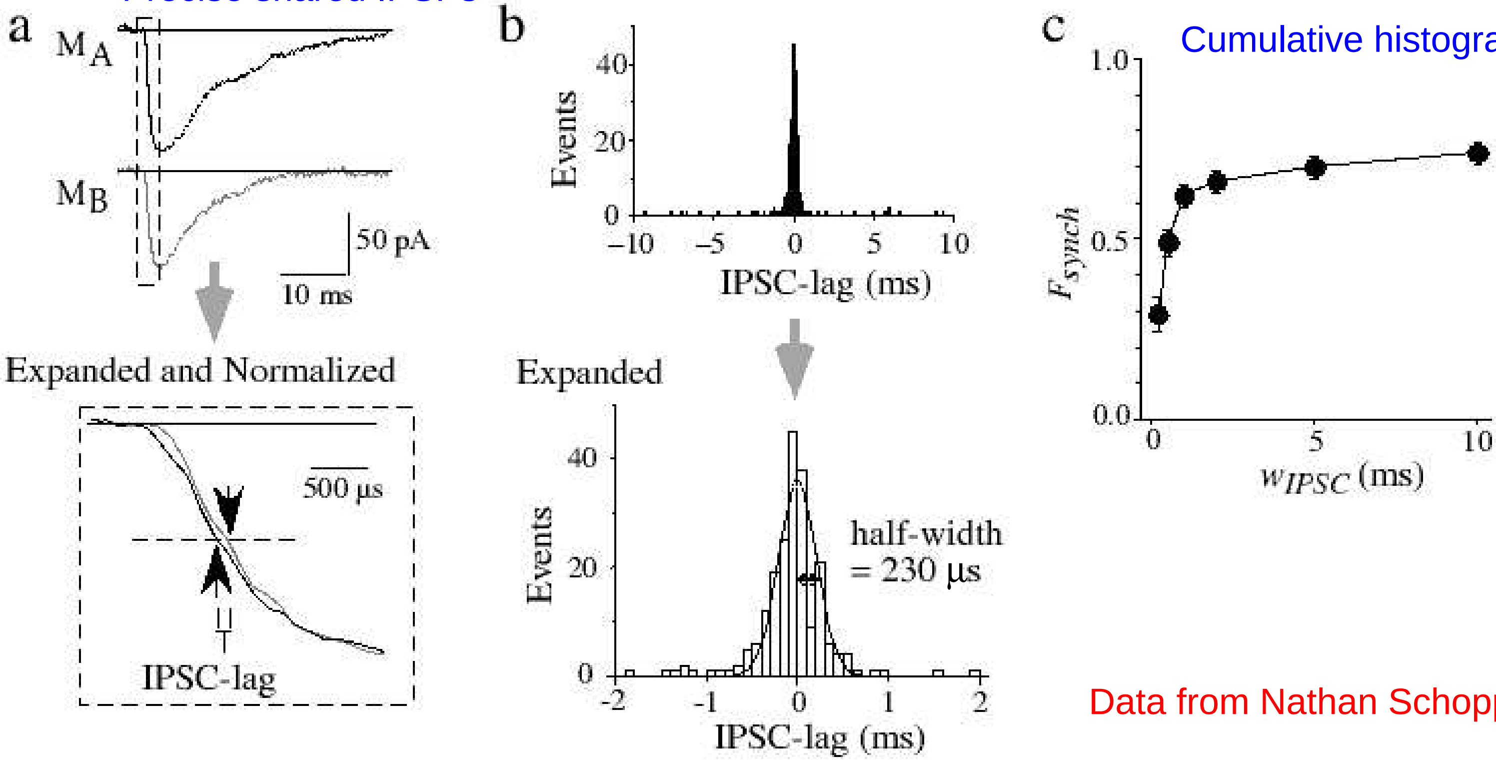
Synchronous transient events

Events are GABA−mediated

Data from Nathan Schoppa

Correlated GC IPSPs

Precise shared IPSPs



Cumulative histogram

Data from Nathan Schoppa

Theory: Poisson - Kuramoto/GBE

- Two oscillators with shared Poisson inputs

$$x' = 1 + \sum_k \delta(t - t_k)\Delta(x)$$
$$y' = 1 + \sum_k \delta(t - t_k)\Delta(y)$$

- This becomes a random map:

$$x_{n+1} = T_n + x_n + \Delta(x_n)$$
$$y_{n+1} = T_n + y_n + \Delta(y_n)$$

- Difference:

$$z_{n+1} = F'(v_n) z_n \quad F(x) = x + \Delta(x)$$

Theory: Poisson, ctd

- Contraction:

$$\rho = \lim_{n\to\infty} \left(\prod_{j=1}^{n} |F'(v_n)| \right)^{\frac{1}{n}}$$

$$\lambda \equiv \log \rho = \int_0^1 P(v) \log |F'(v)| \, dv$$

- But $P(v)$ is uniform and if $F(x)$ is monotone:

$$\lambda = \int_0^1 \log |F'(v)| \, dv \leq \log \left(\int_0^1 F'(v) \, dv \right) = 0$$

from Jensens inequality

- Equality only when trivial PRC, so always contracts!

Theory: White noise

1. Ritt, PRE, Teramae PRL 2004, GBE (back of envelope)
2. Two uncoupled oscillators w/ white noise

$$\begin{aligned} dx &= (1+a)dt + (pdW + (1-p)dW_x)\Delta(x) \\ dy &= dt + (pdW + (1-p)dW_y)\Delta(y) \end{aligned}$$

3. Consider the difference:

$$\begin{aligned} dz &= a\,dt + p\Delta'(u)zdW + (1-p)[\Delta(y)dW_y - \Delta(x)dW_x] \\ &\approx a\,dt + pDzdW + (1-p)GdW_1 \end{aligned}$$

   where $D = ||\Delta'(u)||_2$ and $G = ||\Delta(u)||_2$.

4. Stationary phase distribution:

$$P(z) = K\frac{e^{a \arctan dz/g}}{d^2z^2 + g^2} \quad d = pD, \;\; g = (1-p)G.$$

5. Centered around $a/2d^2$ so it is strongly synchronizing

Further questions

- Advantage of explicit maps over Monte carlo
- Analytic expressions for random maps (yes if $\Delta(x)$ is small)
- More than 2 oscillators?
- Infinitely many - Kuramoto analogue?