The Cognitive Foundations of Mathematics: human gestures in proofs and mathematical incompleteness of formalisms[1].

Giuseppe Longo

Dépt. d'Informatique, CNRS – ENS et CREA, Paris
http://www.di.ens.fr/users/longo

1. Introduction

The foundational analysis of mathematics has been strictly linked to, and often originated, philosophies of knowledge. Since Plato and Aristotle, to Saint Augustin and Descartes, Leibniz, Kant, Husserl and Wittgenstein, analyses of human knowledge have been largely endebted to insights into mathematics, its proof methods and its conceptual constructions. In our opinion this is due to the grounding of mathematics in basic forms of knowledge, in particular as constitutive elements of our active relation to space and time; for others, the same logic underlies mathematics as well as general reasoning, while emerging more clearly and soundly in mathematical practices.

In this text we will focus on some "geometric" judgements, which ground proofs and concepts of mathematics in cognitive experiences. They are "images", in the broad sense of mental constructions of a figurative nature: we will largely refer to the well ordering of integer numbers (they appear to our constructed imagination as spaced and ordered, one after the other) and to the shared image of the widthless continuous line, an abstracted trajectory, as practice of action in space (and time).

Their cognitive origin, possibly pre-human, will be hinted, while focusing on their complexity and elementarity, as well as on their foundational role in mathematics. The analysis will often refer to [Châtelet, 1993], as the french original version of this paper was dedicated to Gilles Châtelet: his approach and his notion of "mathematical gesture" (in short: a mental/bodily image of/for action, see below) has largely inspired this work. However, on one hand, we will try to adress the cognitive origins of conceptual gestures, this being well beyond Châtelet's project; on the other, we will apply this concept to an analysis of recent "concrete" mathematical incompleteness results for logical formalisms. A critique of formalism and logicism in the foundations of mathematics is an essential component of our epistemological approach; this approach is based on a genealogical analysis that stresses the role of cognition and history in the foundations of mathematics, which we consider to be a co-constituted tool in our effort for making the world intelligible. A survey of related approaches to the foundations of Mathematics may be found in [Doridot, Panza, 2004].

2. Machines, body and rationality

For one hundred years, hordes of finite sequences of signs with no signification have haunted the spaces of the foundations of mathematics and cognition and indeed the spaces of rationality. Rules, which are finite sequences of finite sequences of signs as well, transform these sequences into other sequences with no signification. Perfect and certain, they are

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[1] In **Images and Reasoning** (M. Okada et al. eds), Keio University Press, 2005. This paper is the revised version of the english translation, by Pierre S. Grialou, of Longo's part in Francis Bailly, Giuseppe Longo. Incomplétude et incertitude en Mathématiques et en Physique. *In* **Il pensiero filosofico di Giulio Preti**, (Parrini, Scarantino eds.), Guerrini ed assoc., Milano, 2004, pp. 305 – 340.
I am greatly endebted to Pierre for his insightful comments and translation.

supposed to transform the rational into rational and stand as a paradigm of rationality, since *human rationality* is in machines. "Sequence-matching" reigns undisputed: when a sequence of meaningless signs matches perfectly with the sequence in premise of one of the rules (the first at hand, *à la* Turing), it is transformed into its logic-formal consequences, the sequence in the next line; this is the mechanical-elementary step of computation and of reasoning. This step is certain since it is "out of us"; its certainty does not dependent on our action in the world, it is due to its potential or effective mechanizability.

All of that is quite great when one thinks to the transmission and elaboration of digital data, but, as for the foundations of mathematics (and knowledge), a schizophrenic attitude is repeating itself. Man, who has invented the wheel, excited by his genius invention, has probably once claimed "my movement, the movement, is there, *in the wheel*… the wheel is complete: I can go anywhere with it" (well, the wheel is great but as soon as there is a stair…); in this way, he thought he could place, or rediscover, his own movement out of himself. In this way, the lever and the catapult have become the paradigm of the arm and its action (Aristotle). Gears and clocks' strings coincide with body mechanisms, including brain mechanisms (Descartes); and the contraction of muscles is like the contraction of wet strings (Cartesian iatro-mecanicians in the XVIIth century, see [Canguilhem, 2000]).

But some claim the last machine, the computer, has been invented by referring to human beings and their thinking, while it was not the case for the lever and the catapult or the clock mechanisms and their springs. These are not similar to our own body parts, they have not been designed as a model of them. Here is the strong argument of formalists: this time the mathematical proof (Peano, Hilbert), in fact rationality, has been first transferred into something potentially mechanic. Then, on this basis, engineers have produced machines. This is a strong argument, from a historical point of view, but it leaves this functionality of human being (rationality, Mathematical proof), his intelligence, out of himself, out of his body, his brain, his real-life experience. And schizophrenia remains. It has only preceded (and allowed) the invention of the machine, of computer: it is upstream, in the formalist paradigm of mathematical deduction, indeed cognition, since man, in the minimal (elementary and simple) gesture of thinking, would be supposed to transform finite sequences with no signification into finite sequences with no signification, by sequence matching and replacement (from Peano to Turing, see [Longo, 2002/2]; some refer to Hobbes and Leibniz as well).

But where is this "human computer" whose elementary action of thinking would be so simple? As for the brain, the activity of the least neuron is immensely complex: neurons have very different sizes, they trigger large biochemical cascades inside and outside their cellular structure, their shapes and the form of their electrostatic field change; moreover, their activity is never isolated from a network, from a context of signification, from the world. In fact, as for thinking, when we go from one sentence to another one, with the simplest deduction ("if…then…"), we are not doing any sequence matching, but we move and deform huge networks of signification. Machine, with its very simple logical gates, with its software built on even more simple primitives, can only try to functionally imitate our cognitive activities. Machines do not "model" these activities, in a physical-mathematical understanding of modelling - which means to propose a mathematical (and/or artificial) framework able to reproduce the constitutive principles of the object modelled. But even the functional imitation is easy to recognize[2].

On the other hand, some algebraic calculi require purely mechanical processes and are an integral part of mathematics. These calculi, we shall argue, are the death of mathematics,

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[2] The distinction imitation versus modelization is inspired by [Turing, 1950] and explained in [[Longo, 2002/2], where limits to imitation *a la* Turing are explained.

of its meaning and its expressive richness, if they are isolated from contexts of signification and considered as a general paradigm for it.

4. Amoeba, motivity and signification

What else have in common clocks, formalisms, and computers? All artefacts are actually constituted by *elementary* and *simple* components: washers and ropes, 0 and 1 sequences, logical gates, all individually very simple, are put together and associated into huge constructions that may reach a very high complexity. In fact, complexity is the result of a construction which superposes extremely simple constitutive elements: this ability to be reproduced, to be accessible (that is, to be dismantled component by component) is the strength of artificial constructions, which, by this, always proceed "bottom-up". On the contrary, the *elementary* biological component, the cell, is extremely *complex*; it contains all the objective complexity of life; it is elementary since once the cell is cut, it is not living anymore[3]. And embryogenesis is always "top-down".

The contraposition between artificial and natural, just sketched above by referring to the *elementary* and *simple* aspects of artefact as opposed to the *elementary* and *complex* aspects of natural phenomena is the core of our analysis: we will face it again in the complexity of linguistic symbols, of living cells, of strings in quantum physics. At each phenomenal level of this three level classification (human languages, biological entities, microphysics), rough yet historically rich, of the way the world appears to us, the elementary seems extremely complex, perhaps the most complex in the phenomenality and for sure the most difficult to understand.

Moreover, the cell like the amoeba (or a … paramecium, see [Misslin, 2003]), changes internally as well as in its relationships with the external: it moves. This is essential for life, from its action in space to cognitive phenomena since "motivity is the original intentionality" [Merleau-Ponty, 1945]. Now, in our opinion, signification is constituted by the interference of signal with an intentional gesture, be this gesture "original" or not. In this way, gesture, which begins in motor action, set the roots of signification between the world and us, at the interface of both. The chemical, thermal signal, which affects amoeba and cell, is "significant" for the living, regarding its current internal change, its action, and its movement. The neuron, reached by a synaptic discharge which deforms its membrane and its electrostatic field, reacts with a biochemical cascade, with a subsequent deformation of its electrostatic field, even by changing form and place of synaptic connections. In other words: it reacts with an action, a gesture at its scale, with its internal and external mobility; at its level, this reaction is meaning. And the elementary, minimal, living unit is preserved while the current action is modified by the signal (each neuron, as any living, is always being acting). This modification is at the roots of signification, a thesis already sketched in [Longo, 2003]. Of course, the neural network also changes and the net of networks, and the brain as well, in a changing body. This is the modified activity of this entanglement and the awfully complex coupling of organization levels which makes significant the friction between the living and the external world. The result is non-additive, indeed non-compositional (it is not possible to rebuild it by assembling elementary significations "pieces by pieces"), since it requires the activity of the whole network. Finally, perception itself is equivalent to the difference between an active forecast and a signal (this analysis has gone from [Merleau-Ponty, 1945] up to [Berthoz, 1997]). For this reason, perception leads to signification: it depends on prevision, which is an action, and accompanies any other action. Perception is the result of interference between a

[3] This comparison artifact-living introduces an analysis of biological complexity, in [Bailly, Longo, 2003].

signal and an action or an anticipation. In short, there is no meaning without an ongoing action.

As for man, signification includes also action within a communicating community, interaction with others in a symbolic culture, rich of language, gestures and evocations. Thus, intentionality allows signification, on the basis of original intentionality: motion. This very intentionality which envelops any object of thought, as an "aim".

5. The abstract and the symbolic; the rigor

Through this constitutive route, entangled and complex in its origin, which lies on the action of the living, humanity has arrived to propose our symbolic culture. Its symbols are significant, they refer to the world, and they are in resonance with the world. Each symbol, each sentence has a huge "correlation length" in the space of present and history. The correlation space is almost a physical fact: that is why we use this term borrowed from physics. This length describes the possible distance of causal links: each word is almost physically correlated, by an individual or a whole community, to a huge set of words, acts, gestures and real-life experiences. These links constitute a "manifold" in space and time, in a mathematical sense of manifold, since it is possible to give a structure to it (as it has been already undertaken, in the modern analyses of signification spaces, see [Victorri, 2002]). No metaphysics of ineffable, but rather concrete, material and symbolic reality of phylogenetic, ontogenetic and cultural complexity of the human being and his language (see also [Cadiot, Visetti, 2001]). This takes us far away from "thought as a formal computation".

A symbol is thus a synthetic expression of signification links. It can be realized in a linguistic sign or a gesture, as it is a movement and a body posture. Both may be elementary, as they are the minimal components of human expression, but they are very complex since each meaningful sign, each gesture, and in fact each body posture, is the result of a very long evolutionary itinerary and synthesizes it. Once more then, the elementary component of natural phenomenality, the minimal, meaningful symbol of intersubjective communication is very complex. This communication is both human and animal, since the gesture and the body posture are a part of animal expressiveness. And mathematics is *symbolic* and it is meaningful.

In general, a human linguistic symbol (a word, a sentence, a texte) is also an evoked gesture as in the living interaction there is no sign without signification. But this signification relies on relevance and may be multiple: polysemy is in the core of languages and takes part into richness and expressivity of communication ([Fuchs, Victorri, 1996]). Is this compatible with mathematics? We will go back to this point later.

But mathematics is also *abstract*, which is another huge cognitive problem at stake. This abstraction starts from the categorizations of reality, proper to every animal neural system (see [Edelman, Tononi, 2000]), which are based on the independence regarding sensory modality, indeed on multimodality of the sensory-motor loop ([Berthoz, 1997]). Concepts of our cultures are the organized expression of these categorizations. This expression has been built through an intersubjective exchange within language, which stabilizes experience and common categorization. Mathematical abstraction is a part of this, but it has its own character due to the maximality of its practical and historical invariance and stability.

Finally, mathematics is *rigorous*. The rigor of proof has been reached through a difficult practical experience. It probably started in the Greek agora, where coherence of reasoning used to lead the political debate, where a sketching of democracy has given rise to science, in particular to mathematics, as the maximal place of convincing reasoning. Rigor lies in the stability of proof, in its regularities that can be iterated. Mathematical logic is a set of *proof invariants*, a set of structures that are "preserved" from one proof to another or which are preserved by proof transformations. Logic does not precede mathematics, rather it has followed mathematics: it is the result of a distilled praxis, the praxis of proof. Logic is in the structure of mathematical arguments, it is made of their maximally stable regularities. Of course it has been necessary to distil it from a practical experience not always perfect: the richness and confusion of a large part of mathematics in the XIXth century is an example of this imperfection. Norms were necessary: people did not know what giving a good definition could look like: some were using defining their concepts soundly (Weierstrass), but others, not less great, used to confuse uniform continuity with continuity, say (Cauchy). The formalist answer, identifying rigor with *formal* rigor, might have been necessary. Nowadays only are we able to highlight logic in the structure of proofs ([Girard, 2001]) and to keep away from sequence-matching, a mechanical superposition of sequences of signs which is the motor of any formalism[4].

Thus, mathematics is *symbolic*, *abstract* and *rigorous*, as many form of knowledge and human exchange. But it is something unique in human communication, based on these three properties, since it is *the place of maximal conceptual stability and invariance*. This means, no other form of human expression is more stable and invariant regarding transformation of meaning and discourse. In mathematics, once a definition is given, it remains. In a given context, stability forbids polysemy but not meaning. Invariance is imposed to proofs. This can even constitute a definition of mathematics: as soon as an expression is maximally stable and conceptually invariant, it is mathematics. But let us be careful, we use 'maximal' rather than 'maximum' because we aim to avoid any absolute. Moreover, mathematics is a part of human communication and of the tools the man has found in order to organize its environment and make it more intelligible.

In order to escape from the rich confusion of the XIXth century, from the "wildest visions of delirium" proposed by the models of non-Euclidean theories ([Frege, 1884, p. 20]), and from some minor linguistic antinomies, a strong, perhaps too strong, paradigm was necessary in order to establish robust foundations. For the Hilbertian school it was the paradigm of finitary arithmetic as the essence of a logical or formal system. Brave response to the disorder and to the conceptual and practical richness of mathematics of that time. But a reification, almost a parody, of what is maximally symbolic, abstract and rigorous: mathematics. A caricature of these three pillars of human cognition: the symbolic, the abstract and the rigorous. These three notions, very different one from the others, have been identified by formalism with each other and with another notion, quite flat, the *formal* as a "finite sequence of signs without signification, manipulated by possibly mechanisable rules". Such sequences are thus manipulated by very simple formal rules, as the rule that checks and changes a 0 for a 1 (or the opposite), one at once, as in a Turing machine (sequence matching and replacement). Thus the identity

symbolic = abstract = rigorous = *formal,*

constitutes, from the point of view of human cognition, the crowning glory of simplistic thought, indeed just a thought of mechanics.

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[4] For example, a formalist interpretation and a computer use of 'modus ponens', "*from* A *and* A→B *deduce* B", that is its "operational semantics", consists only in controlling in a mechanic way that the finite sequence of signs (0 and 1) which codes for the first A is identical to the sequence which codes for the second A. Following this, "*then*" writes, actually copies, B: this "sequence-matching" is all what a digital computer can do, modulo very simple "syntactic unification" procedures. Within recent Girard systems, only geometric structure of deduction is preserved (in this case a "plug-in" or a "connection"). This geometric proof structure does not separate syntax from semantics and allow managing, like human reasoning, signification networks.

6. From the Platonist response to action and gesture.

Of course, some did not put up with this reifying schizophrenic attitude, the formalism and its fantastic machines. First of all, Gödel, but almost all the major mathematicians who have come after him, from MacLane and Wigner to René Thom or Alain Connes (Von Neumann might be the only exception) have reacted by adopting a more or less naïve Platonism. Concepts and structure of mathematics are "already there", we just have to discover them, to see them (in fact, such great scholars can "see" these concepts and structures, before doing any proof; this belongs to the practical experience of every mathematician and we should study this cognitive performance as such, without transforming it into any ontology). Sometimes, arguments have referred to the first Gödel incompleteness theorem. This was accompanied with a severe misunderstanding of the theorem and has triggered a kind of "gödelite" which has not disappeared yet from the pathologies of the philosophy of mathematics ([Longo, 1999a; 2002]).

We should now rediscover the meaning in proofs, in the act of deduction, as it is the aim of Platonists too, but in our case we aim to do so without any pre-existing ontology. We should rebuild the constitutive background of mathematics, by carrying out an analysis of their cognitive foundations on the basis of the actions and gestures which give rise to signification in humans, as we have just sketched out. This approach will obviously replace, in mathematics and in natural science, the notion of "ontological truth" by *knowledge construction*, ultimate result of the human cognitive activity, as well as, thanks to this activity on reality, the notion of *construction of objectivity*. Our undertaking is a part of such project; in the case of proof, J-Y Girard's approach is at the forefront of this project. Our background in lambda-calculus and constructive proofs ([Girard, 1990]) has played an important role in the approach we have chosen. Despite its formal origins, lambda-calculus codes proofs in a very structured way[5]: it preserves the organization of proofs and as such differs very much from the coding necessary to reduce formal deduction to the elementary steps computed by Turing machines, as well as by other systems for computability. This coding destroys the architecture of deduction and makes it loose its signification. Böhm trees (even Levy-Longo trees, a slight variation of them, see [Longo, 1983]) bring a structure to lambda-terms which would be otherwise too flat. Girard has proposed his ideas, which highlight the geometry of proof and the regularities that manage and transform meaning, on the basis of lambda-calculus (see Girard, 2001). However, the shade of the formalist methods and philosophies of Church and Curry ([Seldin, 1980]) is now very far away. Such a geometry of proof is thus rigorous and it is compatible with a cognitive analysis of the constitution of concepts and of the abstract and symbolic structures of mathematics we have proposed. Now, first of all, these structures organize space and time.

As for the gesture, as it is an elementary, yet complex, action of living, it is at the origin of our relation to space, our attempts to organize it, therefore at the origin of geometry. In this view, it is worth reading Poincaré: gesture and movement "evaluate distance" (see the many quotations from Poincaré in [Berthoz, 1997]). We will focus now on a very ancient gesture, the conscious eye saccade (jerk), which draws the predator chase line [Berthoz, 1997], and on how this may contribute to establishing a mathematical invariant.

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[5] Writing rules of typed lambda-terms are exactly rules of formal deduction. Although everything remains formal, a term allows to see proof structure showing through [Girard et al., 1990]. Links between types and objects of "geometric categories" (in particular, topos) complete the mathematical richness of system (see [Asperti, Longo, 1991]).

6.1 The mathematical continuous line.

The example discussed here refers to a constitutive gestalt of mathematics: the line without thickness (without width, as Euclid would put it). From the cognitive point of view, one can refer, first of all and simultaneously, to the role of:

- the jerk (saccade) which precedes the prey,
- the vestibular line (the one that helps to memorise and to continue the inertial movement),
- the visual line (which includes the direction detected and anticipated by the primary cortex).

The isomorphism proposed by Bernard Teissier (of Poincaré-Berthoz, according to his definition) is one between the latter two cognitive experiences: action and movement impose (they make us perform) an identification (an isomorphism) between the experience of inertial movement, conducted in a straight line, and the forward saccade, which precedes the movement. This isomorphism is to be extended by ocular pursuit, as saccades, also being an action, and it has for result, as a pre-conceptual practice, a pure direction, without thickness. The invariant of these three cognitive practices is a outlined line, a pre-conceptual abstraction, an abstract practice: it is that which counts, what is in common, distilled in the memory of the action, for the purpose of a new action.

This practice confers meaning and is at the origin of (it enables) the conceptual, linguistic and historical baggage by which one manages to propose the continuous line, parameterised on the real numbers in Cantor-Dedekind style. To put it in other words, this line without thickness is the pre-conceptual invariant of the mathematical concept, invariant in relation to several active experiences; it is irreducible to only one of them. This invariant is not the concept itself, but it is foundational and is the locus of meaning: we do not understand what is a line, do not manage to conceive of it, to propose it, even in its formal explicitation, without the perceived gesture, or even without drawing it on the blackboard, without it being felt, appreciated by the body, through that which was evoked by the first teacher.

I believe that it is necessary here to emphasize the key role of memory, in one of its most important characteristics: the capacity to forget. The intentional lapse of memory, as result of an aim (even preconscious, if we accept to broaden husserlian intentionality), is constitutive of invariance: from the selective role of vision (intentional), an active glance, a palpation by sight (Merleau-Ponty), to reconstruction by memory, which intentionally selects (even unconsciously) that which is important, in view of the action. Until the conceptual construction, it is the capacity to forget that which is not important, in relation to the goals in question, which precedes the explicitation of the invariant, of that which is stable in comparison to a plurality of actions-perceptions. Memory selects, by forgetting, and, by this, practices and yields invariance.

But how may one prove this? There is so little work even and simply of a "gestaltist" nature in the foundations of mathematics or even in mathematical cognition! Yet, mathematics organizes the world in the manner of a science of structures: points, isolated and non-structured, are derived, for example, as the intersection of two lines, Wittgenstein tells us, while Euclid actually constructs this. In fact, a line is not a set of points. It is a gestalt. One can rebuild it using points (Cantor-Dedekind), but also can without points (in certain topos by Lawvere, [Bell, 1998]). And its cognitive foundation and understanding should largely rely on a gestaltist approach.

In conclusion, the memory of this gesture is a prior experience toward a very important mathematical abstraction. It is an “abstract” animal experience since it is the *memory of a forecast,* a forecast of a line which is not there and that memory takes out of its context, by forgetting everything “not important”, which is not the object of intentionality, of a conscious or unconscious aim. Memory of a continuous line, since space of movement is connected, a thickness-less line, since the line itself is missing, pure trajectory, this is the pre-conceptual experience of Euclidean lines, as well as of our modern lines which are parameterized on real numbers. Here is one of the constitutive pillars of the knowledge construction we are talking about: it starts from the abstract, categorizing memory of the predator, its memory of actions in space (indeed of their forecast), and goes up to our abstract, mathematical and rigorous concept of a continuous line, a parametrized trajectory, which is given within language. But the meaning of this conceptual construction, which organizes space and knowledge, stems from the very first gesture of the predator, from its original intentionality, as it is an action; it stems from its meaningful interaction with the environment. Thus, an established mathematical construction, a trajectory parameterized on real numbers *à la* Cantor-Dedekind, is meaningful to us since we have this (common) gesture in our constitutive background.

7. Intuition, gestures and the numeric line.

Mathematical intuition precedes and accompanies theories, since it constitutes the profound unity - but this time graspable in action - of a theory; yet, it also follows them, since "understanding [a mathematical theory] is to catch its gesture and to be able to continue it" ([Cavaillès, 1981] quoted in [Châtelet, 1993; p 31; transl.2000, p 9]). Now, intuition may be grounded in gestures, which may evocate images. Indeed, Châtelet as well takes up this role of gesture again: "this concept of gesture seems to us crucial in our approach to the amplifying abstraction of mathematics…gesture gains amplitude by determining itself…it envelops before grasping…[it is a] thought experiment", [Châtelet, 1993; p.31-32; transl. 2000, p9-10]. In mathematics, there is a "… talking in the hands… reserved for the initiates. A philosophy of the physico-mathematical cannot ignore this symbolic practice, which is prior to formalism…" [Châtelet, 1993; p.34; transl. 2000, p 11].

Gesture of imagination should be included in this physical and mathematical intuition as "sense of construction". By doing this kind of gesture, a human performs a conceptual experiment: "Archimedes, in his bathtub, imagines that his body is nothing but a gourd of water…Einstein takes himself for a photon and positions himself on the horizon of velocities" [Châtelet, 1993; p.36; transl. 2000, p 12]. "Gauss and Riemann…[conceive]…a theory on the way of habiting the surfaces" [Châtelet, 1993; p. 26]. Gauss and Riemann intrinsic geometry of curve surface and space is the "delirium" against which the logicist response will fly into a rage (see [Tappenden, 1995]). Later on, the formalists proposed a solution to this "delirium": it consists of meaningless axiomatic systems, controlled by formal rules, whose coherence relies on the coherence of formal arithmetic into which they can be coded ([Hilbert, 1899]). Only arithmetic and arithmetical induction (as logical laws or purely formal rules, it depends on the authors) are supposed to be the foundations of mathematics, and are considered to be the unique place for objectivity and certainty. Then monomania has started: indeed, on its own, number theory and induction are very important and are an essential component of the foundations of mathematics, the problem comes when considering them as the unique basis for mathematics. This focus on language and "arithmetical laws of thought" has given rise to wonderful digital machines but also to a philosophical and cognitive catastrophe which still remains.

But what is this logico-formal induction, ultimate law of thought, considered by Frege as logical and meaningful and considered by Peano and Hilbert to be purely formal, indeed as a calculus to be computed by a machine?

In formal terms, once a well formed and expressive enough language is given (we need 0, successor operator, rules for quantification and few others, all simple and elementary), we can write the following rule for a predicate A of this language:

$$\frac{A[0] \qquad \forall y\,(A[y] \rightarrow A[y+1])}{\forall x\, A[x]}$$

"Categorical" rule for Peano and Frege (although this term was not in use at that time): this means that number theory is contained in this rule and there is nothing else to say; in other words this rule has only one model. But a true "delirium" is going to show up soon: Lowenheim and Skolem will prove that Peano arithmetic, which should have tallied with integers, has models in all cardinalities! Worst: non-standard models (and non-elementary equivalent, in technical terms), consequences of incompleteness, give nightmares to logicists. True pathologies of incompleteness, with a weird structure of order ($\omega + (\omega^* + \omega)\eta$, where ω is the integers' order type, ω^* its inverse, η the rational numbers' order type), they are of no use, except to give alternative proofs of modern incompleteness results (an amusing exercise for the author of these lines, in his young age). On the contrary, if one thinks that the three large classes of Riemannian manifolds which modelize the fifth Euclidean axiom, and its two possible negations as well, have all acquired an important physical meaning, then one should revise the fregean orthodoxy: the delirium is the one of arithmetic axiomatic, of logical induction and its models but surely not the one of geometry.

But going back to the *signification* of induction may avoid such delirium. Here is a first idea: "conceive the indefinite (unlimited?) repetition of an act, as soon as this act is possible once" [Poincaré, 1902; p.41]. This act, this iterated gesture performed in space, is the well order of the potentially infinite sequence of integers. But, what is this sequence? It is the result of an extremely complex constitutive itinerary. It has started from the counting of small quantities and the establishing of correlations between small groups of objects or the ranking of certain objects as well, as we share with many animals, [Dehaene, 1997]. Then, it has developed through our ranking, counting and spatial organizing experiences, which are as old as humans, until the huge variety of their linguistic expressions. Such expressions have often been devoid of any generality and unable to suggest a general concept, since they were based on individual objects (see [Dehaene, 1997] and [Butterworth, 1999] about some kinds of enumeration in peoples with no written languages). Perhaps it is only possible to isolate the concept of number once writing has stabilized thought, although this is not immediate. Sumerians used different notations to refer to 5 or 6 cows and 5 or 6 trees. Should we claim they used to have a general concept of integers? It is doubtful. Sumerians and Egyptians as well, have reached much later a uniform notation, independent of the numbered object. Greek mathematics has followed, with a beginning of number theory (that is, quite general, invariant and stable concepts, like that of prime number for instance, which does not depend on representation – it is invariant w.r. to the notation). Yet, a full flavour of our theory of numbers needed the establishing a truly abstract and uniform notation for numbers, of any size, as in Chinese, Indian and Arabic cultures.

But a further understanding of the mathematical practice of numbers requires to go back to our discussion on the continuous line. As a matter of fat, starting from little counting by animals, the SNARC effect described in [Dehaene, 1997] seems to distribute integer numbers on a mental line. This is a cognitive bulk which we see, for example, behind an elementary judgement (irreducible to a finitary formalism), like the well-order of the integers (mathematically: "a nonempty generic subset of integers contains a smallest element"), a result of the ordering of numerical practices on a line (see [Longo, 2002] where we discuss of its role in certain recent incompleteness results for Arithmetics). Because the structured order of the integers also participates in the direction of the movement, of the gesture which arranges them on a line, this gestalt which remains in the background, but which contributes to organising them, to arranging them, to "well-ordering them" towards the infinite, in a highly mathematised conceptual space. Moreover, it is not excluded that in order to grasp the statement of the well-order, so complex albeit elementary, a statement which, in a certain sense, is finitary, one may need control over the whole line, and therefore over projective geometry (or over perspective in painting). Here is this network, constitutive of mathematics, as a structured discipline, which participates in the proof, by necessitating the use of complex gestalts even in numerical theory.

So there we have the immense cognitive (and historical) complexity of an elementary judgement, the well-order of the integers (of which the ad hoc reconstructions by large ordinals are also very complex, see §.7), which transforms the proof not into a chain of formulas, but into a geometry of meaningful and complex correlations, of references, threads relating the mathematical reasoning to a plurality of acts of experience, conceptual and pre-conceptual, as well as to other already constituted mathematical structures. The strength of reasoning, even its certainty, thus lies in its remarkable stability and in its invariance (not an absolute), as a uniformity of deductive methods which is dynamic tough (as changing throughout history), but also in the richness of the lattice of connections which hooks it to a whole universe of practices and of knowledge, even pre-mathematical, even non-mathematical

By this, the concept of integer, reached through language, goes back to space again, since number is an "instruction for action", counting and layout in space: a gesture that organizes mental space, the one of "numeric line" what we all share, [Dehaene, 1997]. This line may be funny in the layman imagination: it swings, it is finite … though this is not the case for the mathematician nor anybody with a (even short) background in mathematics. A mathematician can "see" a discrete, growing numeric line, continuing from left to right (in our culture at least, [Dehaene, 1997]) with no limits. This line thus results in the last step of a long itinerary, which leads to reconsider in the space, a mental space, the concept of number as a generalized iteration or invariant constituted by an iterated gesture in the space of action. On the other hand, the origin of this concept lies also in a temporal iteration. An idea from Brouwer, creator of intuitionist mathematics, can be used to support this hypothesis: phenomenal time is defined as discrete sequence of momentums, as a "partition of a moment of life into two distinct things, one giving the place to the other, yet remaining in the memory" [Brouwer, 1948]. In this view, the mathematical intuition of the integers sequence would rely on the subjective and discrete sequence of time; then computation would be the development of process through a discrete temporality.

In our view, the concept of number and the discrete numeric line, which structures it, are the invariants constituted by the plurality of acts, experienced in space and time. These invariants owe their independence, which characterize intersubjectivity (since this is the experience shared with others that gets the most stable), to language and writing. Now, the inspection of the discrete numeric line, an image built in our mental spaces, is a mathematical praxis of a huge complexity; it summarizes a conceptual history which starts from the counting of small quantities, goes up to modern mathematical practice and founds it.

8. Mathematical incompleteness of formalisms

The Incompleteness Theorems tells us that the structured meaning of number, its numeric line, is elementary, though very complex. More precisely, it is impossible to capture (and break down) with elementary and simple formal axioms, a statement such as "a generic non-empty set of integers has a smallest element" or some of its consequences. Here is the formal incompleteness of formalisms.

In order to understand this mathematical statement, a "geometric judgement of well order", you should leave this page and, once more, have a look at the discrete and growing numeric line you have in your mind. Hopefully, you can see the well order property: first, isolate a generic nonempty subset (generic means here that it has no specific properties, except possessing an element, and thus that is not necessary to define the set explicitly). Then, since this subset contains a number, it contains also a smallest number… In order to see it, look at the existing element, call it p, in your nonempty set; then, there surely is a smaller, possibly equal one, in the set. Look at the smallest, i.e. at the first one within the finite part of the number line that precede p: can't you see it? It is there, even if you cannot and you do not have to compute it. This is the judgement expressed millions of times, by mathematicians (not logicians, of course) using induction in a proof.

Of course, well order implies formal induction, but is much stronger. In fact, well order is the "construction principle" at the core of number theory and formal induction, which is a proof principle, cannot capture it; this is the mathematical incompleteness of formalisms: some formal statements of Arithmetic can be deduced by this judgement, but not by formal induction. Generally speaking, it is possible to describe this mathematical incompleteness as a gap between (structural) *construction principles* and (formal) *proof principles*: this is the point, which may be developed by focusing also on the example of the continuum, see [Longo, 2002; Bailly, Longo, 2005].

Now, it is impossible to understand this story through the proof of Gödel's incompleteness theorem, which is "only" a fantastic undecidability theorem; however, it becomes clear in the proofs of more recent concrete incompleteness theorems (see [Longo, 2002]; this paper includes a technical discussion[6]). Some persist to "force" formal induction in order to prove these theorems. It requires a technically extraordinary difficult *ad hoc* construction, which forces induction all along ordinals, far beyond the countable or predicative (see [Longo, 2002] for some references). Nevertheless, the only uniform method remains a concrete reference to numeric line; moreover, this method is included in the background of any non-logician mathematician. Such a mathematician understands and uses induction in the following way: "the set of integers I am considering is non-empty, therefore it includes a smallest element". That's all and this is cast-iron. Once out of the fregean and formalist anguish, this means out of the forbidden foundational relation to space and out of the myth according to which certainty relies only on sequence matching, then any work based on the ordered structure of numbers, on the geometric judgement lying at the core of mathematics, can go smoothly. Incompleteness show that this judgement is elementary (it cannot be further reduced), but it is still a (very) complex judgment[7].

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[6] Let us go back to the above exercise on the numeric line. The calculability of the first element in the non-empty set will depend (on the level) of the definability of the considered sub-set. In some recent "concrete" examples, we prove that, in the course of a proof, we use a sub-set of integers whose definition, although rigorous, cannot be given in first order, formal arithmetic, which is the place for effective computability. For this reason, such first element (if it exists) is far from being calculable. But humans (with a background in mathematics of course) can understand very well the conceptual construction and the rigorous, though non formal, proof without any need of a ontological miracle; in such a way, we can prove theorems which are formally unprovable (see the next footnote and [Longo, 2002] where normalization and Kruskal-Friedman theorems are discussed). If computers and formalists philosophers cannot do it, this is their problem.

[7] Any strictly formalist approach also rejects some principles which are much less strong than the latter, for such approach rejects even non perfectly "stratified" (predicative) formal systems: the elementary must be absolutely simple and must not allow any "complexifying loop" (self-reference). But, impredicativity is ubiquitous in Kruskal-Friedman theorem, KF (see [Harrington et al., 1985], in particular Smorinsky's articles). It is the same for "normalization theorems" in formal, though impredicative, type theory (the F system [Girard et al., 1990], which has played a very important role in computer science). In facts, its proof through formal induction would require a transfinite ordinal, far beyond the conceivable (but the analysts of ordinals are prepared to do anything,

The natural dimension of mathematics, or in other words its blend of artificial and natural, is built on that: mathematics surely includes some completely and uniformly axiomatizable fragments that can be captured by elementary, *simple* and mechanizable principles (this is the most boring part of mathematics which is now being transferred to computers); however mathematics are also based on complex and elementary judgments, such as the geometric judgment of well order which completes induction and provides it geometric foundations. But does the use of this judgment make the notion of proof undecidable[8]? This is a problem for the machines but not for humans: in fact, the geometric judgment of well order has a "finite" nature and is quite effective from the point of view of numeric line (only a finite initial segment is to be considered, though not necessarily computable: this is the segment which precedes an element of the non empty subset considered). This line belongs to the human mental spaces of conceptual constructions which are the result of action in space and common linguistic experience and, at the same time, of the (spatial) reconstruction of phenomenal time. It is objective and efficient, as any mathematics, because of its constitutive background which fixes its roots in the relation between the world and us. The (very reasonable) effectiveness of mathematics comes from its blend of formal calculi and meaningful naturality.

9. Iterations and closures on the horizon

But what is this finite, so important to formalisms and machines, since it defines the computable, the decidable? Actually, it is not possible to define it formally. Another astonishing consequence of the incompleteness of logico-formal approches is that there is no formal predicate that could determine the finite without having to determine the infinite too. In short, it is not possible to isolate the collection of standard integers, without an axiom or a predicate for the infinite; in other words, formal arithmetic cannot talk about (standard) finite numbers. To do so it is necessary to use a version of set theory including an axiom for infinite. (An analogue situation exists in category theory, more precisely in Topos with natural number object.)

Here is another way to understand why it is not possible to overlook the "numeric line" (or an axiom for infinite): the concept of integer is extremely complex, it is necessary to immerse it into a more rich structure, whether the well ordered space of the sequence of numbers or infinite sets, in order to grasp it. Nevertheless the difference between both kinds of structure is clear: in contrast with our approach, set theory is ontological or formal. In the ontological case, objectivity and certainty are guaranteed by God (which is certainly, for some people, very reliable), while in the formal case they depend on the formal coherence of the theory. Now, the only method to formally prove coherence is a proof done within the

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provided not using the geometric judgement of well order; others, the predicativists prefers to throw the system itself through the window: such monstrous ordinal would confirm it is not "founded"). Another formal analysis of normalization, relevant for the computer-aided proof, prefers to use IIIrd order arithmetic; but… by which theory is the coherence of the latter guaranteed? By the IVth order arithmetic and so on (likewise if a formal set theoretic framework is chosen). In short, the "classic" proof of KF mentioned above uses, in a crucial deductive passage, the geometric judgement of well order, regarding a supposed non-empty and highly non-calculable sub-set (__1 in technical terms). The formal proof of normalization, and the "meaningful" one as well, uses, de facto, the same judgement as the only guarantee of coherence, indeed of sense, and both reject any infinite regression (see [Longo, 2002] for a compared analysis of the provability of these major results of contemporary logic).

[8] It is possible to characterize formal hilbertian systems, in a very wide sense, as deductive systems in which the notion of proof is decidable, in the sense of Turing machines.

framework of a formal set theory including an axiom of infinite, for a bigger infinite cardinal number… thus, the formal coherence game is going on "in perpetuum", as detached, regarding the world, as Platonist ontology. In contrast, the geometric judgement of well order we focused on, is based on a phenomenal real life experience which has started out of mathematics and in particular out of number theory: foundations of mathematics are provided by the cognitive origin of this judgement, by its phylogenetic history based on the plurality of our modes of access to environment, space and time within the framework of intersubjectivity and language. This judgement is to be added to formal proofs of coherence, which are sometimes very informative, like normalization theorems (see [Girard et al., 1990], [Longo, 2002]) , and allows to stop infinite foundational regressions.

Therefore, within mathematics it is impossible to avoid using the infinite in order to talk about the finite. In reference to terms from physics, we could say that finite and infinite are formally entangled. But the current infinite is a "horizon": we can understand it as a limit to the numeric line, as the vanishing point of projective geometry or of the paintings of Piero della Francesca who is one of its creator. Gilles Châtelet formulates this very nicely: "With the horizon, the infinite at lasts finds a coupling place with the finite" [Châtelet, 1993; transl. 2000, p 50]… "An iteration deprived of horizon must give up making use of the envelopment of things" [Châtelet, 1993; transl. 2000, p 52] … "Any timidity in deciding the horizon tips the infinite into the indefinite … It is therefore necessary, in order to refuse any concession to the indefinite and to appropriate a geometric infinite, to decide the horizon" [Châtelet, 1993; transl. 2000, p52]. This is what Piero has done by prospective in his paintings and what mathematicians, at least since Newton and Cantor, do everyday.

In contrast, a machine iterates since "finitude fetishizes iteration [Châtelet, 1993; transl. 2000 p.51)" : one operation per nanosecond without any weariness nor boredom. Here is the difference since, in such situation, humans (and animals) are bored. After a couple of iterations, we get tired and we stop or say: "OK, I got it" and we look at the horizon. This is the true "Turing test", as boredom should be added in order to better test the human-machine difference (see [Longo, 2002/2] for other arguments).

10. Intuition

So far, we have been little discussing about intuition. This word is too rich of history to deal easily with it. Too often, "balayé sous le tapis" it ends in the black holes of explanation. Rigor has been quite fairly opposed to it, up to the "rigor mortis" of formal systems. Many errors in proofs, in particular during the XIXth century, have justified such process (we already mentioned Cauchy's mistakes; but also Poincaré's in his first version of the three bodies theorem should be quoted…; others used to "look" at the continuous functions and to claim they were all differentiable, from right or left- Poinsot). But especially, non-Euclidean geometries delirium had broken "a priori" geometric intuition, ultimate foundation of Newton's absolute Euclidean spaces, in their Cartesian coordinates.

Now, the foundational analysis of mathematics we are developing does not imply the acceptation of whatever "intuitive view". On the contrary, selection must be rigorous, and justification must propose a constitutive analysis of a structure or a concept. In fact, intuition itself is the result of a process which precedes and follows conceptual construction; intuition is dynamic, it is rich of history. After Cantor, for example, a mathematician cannot have the same intuition of the phenomenal continuum as before: he even has some difficulties to view it in a non-cantorian way.

The dialog with sciences of life and cognition allows to reduce the reference to introspection, which was the only tool used by previous analysis in this direction (Poincaré, Enriques). Like any scientific approach, our analysis tells a possible constitutive story, to be confirmed, to be refuted or to be revised: any knowledge, any science, must be strong, motivated, and methodical: however, it remains as uncertain as any human undertaking. Cognitive science analyses, and by doing so, calls into question the very tools of thought: it must then propose a scientific approach, which would be the opposite of a search for certainty lying in the absolute (i.e. non-scientific) laws of logicism.

The two examples which have been proposed here could be considered as paradigmatic, because of their important differences. The reflection which has been carried out above, about geometric judgement of well order (discrete numeric line), is based on one century of work on arithmetic induction and on the quoted cognitive analyses ([Dehaene, 1997], [Butterworth, 1999]; "Mathematical Cognition", which focus on numeric deficits and performances has become a discipline and a journal). The essential incompleteness of formal induction thus indicates the huge mathematical soundness of a common praxis which is an aspect of proofs. As we have said, this praxis is a part of proofs, since, when induction on ordinals or orders (of variables) is forced, it always requires using a further ordinal or a higher order whose justification is not less doubtful. Descriptions of ordinals and orders within set theory similarly leads to an infinite piling-up of the absolute universes we just talked about. On the other hand, the infinite conceptual regression stops at our geometric judgement of well order: if one wants to know what is going on in number theory, there is, at the moment, no other way. This corresponds to the feeling of any mathematician and this is expressed, usually, by a Platonist attitude: the number line is there, God given. Let us change then this pre-existing ontology, concepts without human conceptor, for an analysis of human construction of knowledge.

Poincaré and Brouwer may have opened the way, but the technical developments have followed only the ideas of the latter. However, these developments have undergone, on the one hand, a complete loss of any sense because of the formalization of intuitionist logic by Heyting and his successors (see [Troelstra, 1973]) and on the other hand, the philosophical impasse of the Brouwerian solipsism and language-less mathematics (see [van Dalen, 1991]): such ideas are completely opposed to the constitutive analysis we propose here, which refers, in a fundamental way, to the stabilization of concepts occurring within shared praxis and language in a human communication context. After one hundred years of reflections on the topic, mathematical experience and cognitive analyses suggest how to go back to the practice of inductive proofs and how to use it as a cognitively justified foundation of deduction. In such case, intuition comes at the end of a process which includes also a practice of proof; in short, intuition follows the construction of the discrete numeric line and allows (and justifies) geometric judgement.

The other example introduced above, the memory of a continuous trajectory (§ 6.1), uses some ideas sketched in [Longo, 1997 and 1999] and comes from recent remarks in cognitive science (neurophysiology of eye saccades and pursuit in [Berthoz, 1997]). However, it does not propose any foundation to proof. It consists just of a reference to signification which precedes and justifies conceptual construction, on the basis of its pre-conceptual origin. From Euclid's viewpoint on the phenomenal continuum to Cantor and Dedekind's rigorous construction, mathematics has managed to propose (to create) a continuous line from the abstract trajectory already practiced in the human activities and imagination: in this case, intuition precedes mathematical structure and then is enriched and made more precise by the latter. A mathematician understands and communicates to the student what the continuum is by gesture, since "behind" gesture both share this ancient act of life experience: the eye saccade, the movement of the hand. With gestures and words, a teacher can (and must) introduce to "the talking in the hands…reserved to initiates" what Châtelet talks about. The conceptual and rigorous *re*-construction is obviously necessary: the one of Cantor and Dedekind is one possible example (see the works of Veronese around the late XIXth century or [Bell, 1998] for different approaches), but teaching must also make the student feel the experience of intuition, the experience of "seeing", which lies at the core of any scientific practice.

In this example, the original intuition may not be essential to proofs, however it is essential to comprehension and communication and in particular to conjecture and invention of new structures. Indeed, here lies the serious lack of logicism and formalism, which are entirely focused on deduction: the analysis of the foundations of mathematics is not only a problem of proof theory, but it is also necessary to analyse the constitution of concepts and structures. Set Theory has accustomed people to an absolute Newtonian universe where everything is already said, one just has to make it come out with the help of axioms. On the contrary, mathematics is an expanding universe, with no "pre-existing space", to which new categories of objects and transformations are always being added. Relative interpretation functors allow reconstructing dynamic unity or correlations between concepts and structures, new and ancient.

The two examples which have been studied may seem modest. Nevertheless, they may play a paradigmatic role: the concept of integer number and its order, and the continuous structure of a thickness-less, one-dimensional line, are two pillars of mathematical construction (as already mentioned, the construction of a point without dimension in Euclid is given as intersection of two one-dimensional lines; this is also a way to grasp the previously notion of point). Of course, other examples should be analysed: the richness and "open" nature of mathematics requires analyses of greater richness. For exemple, we should investigate the reconstruction of borders and outlines, that often do not exist, as in some experiments of Gestalt theory. This is another pre-conscious practice of a preliminary form of abstraction (probably at the very low level of the primary visual cortex!). From "Kanitza triangles" and many other analyses (see [Rosenthal, Visetti, 2003]) to works in neurogeometry (see [Petitot, Tondut, 1999]) we are going to understand the richness of the activity of visual (re-) construction. Since vision is far from being a passive perception, it is rather a "palpation through looking" (Merleau-Ponty), it participates into a structuring and a permanent organizing of the environment. We extract, impose, project… forms, which is a kind of pre-mathematical activity we share with at least all animals which are equipped with a fovea and a visual cortex (almost) as complex as our. The friction between us and the world produces mathematical structures, from these elementary (but often complex) activities, up to language and concepts, as soon as this "friction" involves a communicating community.

The analysis of proof also requires this kind of investigation, since no proof of importance comes without the invention of a new concept or a new structure (and this is what is really worth in mathematics). Besides the incompleteness of arithmetic, the one of formal set theory, to whom statements such as the continuum hypothesis escape, shows that even a posteriori formal reconstruction is often impossible. Finally, the scientific approach to intuition should also be built in order to think about the teaching of mathematics. The usual way to teach mathematics, as "an application of (formal) rules", is a punishment for any student and may have helped to the current decreasing of mathematical vocations. It is urgent to go back to signification, to the motivated construction, in order to recover and communicate the pleasure of mathematical gesture.

11. Body gestures and the "cogito"

Complex gesture (which may be non-elementary) evocated by Gilles Châtelet, helps to understand what is at stake. However it is necessary to "naturalize" this gestures much more than what has been done by Châtelet. Indeed, a limit to his thought lies in a refusal of the animal real life experience which precedes our intellectual experience, and also in the lack of an understanding of the biological brain as a part of the body. This body allows gesture among humans, not only in a historical but also in an animal dimension. This is precisely the sense of human we need to grasp. The absence of any understanding of human, regarding its natural dimension, is the biggest mistake of the great and rich turn of the philosophy which greatly owes to the Cartesian "cogito". This turn has formed a gap between us and our real animal life experience in space and time, and has directly leaded to the myths of machine or to ontologies exterior to the world. If only modern philosophy had started with "I am scratching my nose and my head and I think, therefore I am", we would have progressed better. But scratching is not the most important and, provocation aside, what is especially to be grasped is the role of prehension and kinestheses ([Petit, 2003]) in the constitution of our cogitating human mind, starting with the consciousness of our own body, then of the self in its relation with others, up to the explicit thinking in language. In such an undertaking, there would be a reference to a wide gesture which makes us conscious of our body and, through action, places this body in space. The reflexivity/circularity of the abstract and symbolic thinking of self would find there its elementary, though very complex, explicitation, in a thinking of a deduction which would have, as a consequence, at once the thought itself, the thinking of self and the consciousness of being alive within an environment. Husserl describes a strong gesture, another original act of consciousness, the one of a man who "feels one of his hands with the other one" ([Petit, 2003], [Berthoz, Petit, 2003]). This biological, material hand is the very first place for gesture; humanity, actually the human brain wouldn't exist without the hand. This is not a matter of metaphors but rather a concrete reference to what we have been taught by the evolution of species: in the course of evolution, the human hand has preceded brain and has stimulated its development ([Gould, 1977]). In fact, the nervous system is the result of the complexification of the sensori-motor loop. The human brain is what it is because human hand is for this loop the richest possible tool, the richest possible animal interaction in the world. Then, socialization and language (because of their complexity and expressivity) and history as well, have done the remaining work, up to mathematics.

What is lacking to formal mechanisms, or in other words their provable incompleteness, is a consequence of this hand gesture which structures space and measures time by using well order. This gesture fixes in action the linguistic construction of mathematics, indeed deduction, and completes its signification.

12. Summary and conclusion

Why these reflections in a book on basic cognition? As we said at the very beginning, mathematics has always been an essential component of the theories of knowledge. So there lie, in our opinion, the reasons of this focus, reiterated throughout the course of history: they are due to the anchoring of mathematics in some of the fundamental processes of our interaction with the world. The gesture which traces a trajectory, an edge, the following of a prey by eye jerks (saccades), the memory of these gestures, as well as counting to keep, divide, compare... are among some of the most ancient acts carried out by the living beings we are and they participate in mathematical construction. It even seems that writing began with the quantitative recording of debts, by the Sumerians (see [Herrenschmidt, 1996]). In short, it may be that the first great conceptual invariants have been proposed in their specifically mathematical maximality, as developments, complex and constituted through human communication, of the most fundamental originary organising gestures for space and action in space. The cognitive sciences and mathematics have all to gain from a two-way interaction, through the examination of the great problems of the foundations of mathematics.

In particular, we proposed to analyse two features of mathematical reasoning, namely: the construction of mathematical concepts and the structure of mathematical proofs, in order to rediscover sense in mathematics. The idea is that signification is based on language first, but also on gestures, as forms of action and communication, in a broad sense, it actually originates by interference with action; moreover, we point out that linguistic symbols, which are essential to intersubjectivity as locus for human abstraction, are grounded as a last resort on gestures. Yet, mathematical concepts and proofs are developed within language, as a need of intersubjectivity in the context of communication; and the linguistic framework brings further stability and invariance to mathematical concepts and proofs, in particular since writing exists.

In our perspective, thus, the signification of concepts and proofs rely also, in contrast with a Platonist and a formalistic view, on some particular features of human cognition. These features precede language, resort from action and ground our meaningful gestures. We gave two examples of them: the complex constitution of the integer number line and trajectories, from eye saccades to movement. The first grounds the notion of well ordered number line, a constructed mental image, on which the principle of induction used in mathematical proofs relies. Similarly, the image of the continuous line/trajectory is the result of a variety of gestures, including eye saccades. It founds and gives meaning to the subsequent mathematical conceptual (linguistic) construction.

The proof itself, as a particular case of deductive reasoning, relies on gestures: this a consequence of the sketchy analysis developed here of recent "concrete" incompleteness theorems, which show exactly where formal induction is insufficient. Thus, gestures may be involved in mathematics and proofs at different levels. In suggesting and grounding the constructions of mathematical structures and in proofs, by completing the principle of induction, as in the example. But also in the deductive structure of the proof itself: the geometry of proof developed by J-Y Girard, as a new paradigm of deduction (in contrast with 'sequence matching', the tool of formalism) could be used to uncover the organizing structures involved in proofs. In this view, the explicit deductive process, as a result of an exigency of communication, is constrained by language but implicitly involves structured gestures, similarly as concept formation.

The cognitive foundation of fragments of the mathematical practice hinted here is clearly an attempted epistemological analysis, as genealogy of praxes and concepts, in language and before language. The historical construction of mental images is a core component of it, as a key link to our relation to space and time and a constitutive part of our ongoing attempt to organise the world, as knowledge.

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